Jour Pure Math
Vol 6 (1987—88) PP 1—8

INVERSE Γ—SEMIGROUP

N. K. SAHA AND A. SETH

Intro duction :

In [1] Sen and Saha defined a Γ—semigroup, regular Γ—semigroup and studied some properties of regular Γ—semigroup The aim of this paper is to introduce inverse Γ—semigroup. Along this way, we deduce necessary and sufficient condition for a Γ—semigroup to be inverse Γ—semigroup. We show that homomorphic image of an inverse Γ—semigroup is an inverse Γ—semigroup.

1. Preliminaries.

Definition :

Let $M = \{a, b, c, \ldots\}$ and $\Gamma = \{x, y, z, \ldots\}$ be two non empty sets. M is called a Γ —semigroup if (i) $axb \in M$ and (ii) $(axb)\,yc = ax\,(byc)$ for all $a, b, c \in M$ and for all $x, y \in \Gamma$. A Γ —semigroup M will be denoted by $M(\Gamma)$. Since the associativity of the expression $(axb)\,yc = ax\,(byc)$ extends to all expressions of any length, henceforth we omit the bracket in the deduction of proofs of various statements.

Definition :

Let M be a Γ — semigroup. A non-empty subset B of M is said to be a Γ — sub semigroup of M if $B \Gamma B \subset B$.

Definition :

A right (left) ideal of a Γ — semigroup M is a non-empty subset I of M such that

$I \Gamma M \subset I$ ($M \Gamma I \subset I$). If I is both a right ideal and left ideal of M then we say that I is an ideal of M.

Definition :

Let M be a Γ – semigroup. An element $a \in M$ is said to be regular if $a \in a \Gamma M \Gamma a$ where $a\Gamma M\Gamma a = \{ axbya : b \in M \text{ and } x, y \in \Gamma \}$. A Γ – semigroup M is said to be regular if every element of M is regular.

Definition :

Let M be a Γ –semigroup. An element $e \in M$ is said to be an idempotent in M if there exists an $x \in \Gamma$ such that $exe = e$. In this case we say e is an x-idempotent.

Definition :

Let M be a Γ – semigroup and M_1 be a Γ_1—semigroup. A pair of mappings $f_1 : M \to M_1$ and $f_2 : \Gamma \to \Gamma_1$ is said to be a homomorphism from (M, Γ) to (M_1, Γ_1) if $(axb) f_1 = (af_1)(xf_2)(bf_1)$ for all $a, b \in M$ and $x \in \Gamma$. If f_1 and f_2 are both surjective then (f_1, f_2) is said to be a homomorphism from (M, Γ) onto (M_1, Γ_1).

Definition :

Let M be a Γ –semigroup. Let $a, b \in M$ and $x \in \Gamma$ If $axb = bxa$ we say that a, b are x–commutative. A Γ –semigroup M is said to be commutative if $axb = bxa$ for all $a, b \in M$ and for all $x \in \Gamma$.

Definition :

Let M be a Γ –semigroup and $a \in M$. Let $b \in M$ and $x, y \in \Gamma$. b is said to be an (x,y) inverse of a if $a = axbya$ and $b = byaxb$. In this case we shall write $b \in V_x^y(a)$. We may also write $b = a^{-1}_{x,\, y}$.

If $b \in V_x^y(a)$ then a and b are necessarily regular elements of M. On the other hand let a = axcya where c ϵ M and x, y ϵ Γ be a regular element of M. Let b = cyaxc. Then axbya = a and byaxb = b. Thus if a = axcya be a regular element of M then b is an x, y) inverse of a where b = cyaxc.

2. Inverse Γ – semigroup

Definition :

A regular Γ – semigroup M is called an inverse Γ – semigroup if $|V_x^y(a)| = 1$, for all a ϵ M and for all x, y ϵ Γ whenever $V_x^y(a) \neq \phi$. That is every element a of M has a unique (x, y) inverse whenever (x, y) inverse of a exists.

Theorem (2, 1) :

Let M be a Γ—semigroup. M is an inverse Γ – semigroup if and only if (i) M is regular and (ii) if e and f be any two x-idempotents of M then exf =fxe, where x ϵ Γ.

Proof. Suppose M is an inverse Γ —semigroup. Then by definition M is regular. Next let e and f be two x-idempotents of M. Let a ϵ V_y^z (exf). Then exfyazexf = exf . . . (1) and azexfya = a . . . (2) Now fyazexfyaze = fyaze by (2). Therefore fyaze is an x-idempotent . . . (3). Also exfxfyazexexf = exf by (1) and fyazexexfxfyaze = fyaze by (3). Hence exf ϵ V_x^x (fyaze . But fyaze being x-idempotent belongs to V_x^x (fyaze). Therefore exf = fyaze since M is an inverse Γ – semigroup. Hence exf is an x-idempotent. Also, fxe is an x-idempotent of M. Indeed, if b ϵ V_u^v (fxe), then fxeubvfxe = fxe and bvfxeub = b. Then eubvfxeubvf=eubvf Thus eubvf is an x—idempotent. Also, fxexeubvfxfxe = fxe and eubvfx-fxexeubvf = eubvf. Hence fxe ϵ V_x^x (eubvf). But eubvf being x-idempotent belongs to V_x^x (eubvf). Hence fxe = eubvf Thus fxe is an x-idempotent of M. Now exfxfxexexf = exfxexf = exf and fxexexfxfxe = fxexfxe = fxe. So, fxe ϵ V_x^x (exf). But exf ϵ V_x^x (exf). Therefore, exf = fxe.

Conversely let (i) and (ii) hold. We shall prove that M is an inverse Γ– semigroup. Let b, c $\in V_x^y$ (a) where a $\in$ M. Then axbya = a and byaxb = b . . . (4). axcya = a and cyaxc = c . . . (5). Now each of axb and axc is y-idempotent of M and each of bya and cya is x-idempotent of M. Also axbyaxc = axcyaxb using (ii). Therefore axc = axb. Similarly byaxcya = cyaxbya. Therefore bya = cya. Then b = by (axb) = by (axc) = (bya) xc = (cya)xc=c. Thus M is an inverse Γ — semigroup. This completes the proof.

We remember the following definitions :

Definition : Let A and B be two sets. Then any subset of A $\times$ B is called a binary relation from A to B.

Definition : A binary relation f from A to B is said to be an one-one partial transformation from A into B if the following conditions hold

(i) (x,y) $\in$ f, (x,y$'$) $\in$ f implies that y = y$'$

(ii) (x$'$,y) $\in$ f, (x, y) $\in$ f implies that x = x$'$

The set of all one-one partial transformations from A into B will be denoted by **J** (A, B).

Lemma (2.2) : **J** (A,B) is a Γ-semigroup, where Γ = **J** (B,A).

Proof. Let f, g, h, . . . belong to **J** (A,B) and let α, β, γ, . . . belong to **J** (B,A). We define fαg = { (a,b) $\in$ A$\times$B : there exists $a_1 \in$ A, $b_1 \in$ B such that (a,b_1) $\in$ f, (b_1,a_1) $\in \alpha$, (a_1, b) $\in$ g }. Obviously, f α g is a binary relation from A to B. Let (x, y) $\in$ fαg and (x, y$'$) $\in$ fαg. Now (x,y) $\in$ fαg implies there exist $a_1 \in$ A, $b_1 \in$ B such that (x,b_1) $\in$ f, (b_1,a_1) $\in \alpha$, (a_1, y) $\in$ g and (x,y$'$) $\in$ fαg implies there exist $a_2 \in$ A, $b_2 \in$ B such that (x,b_2) $\in$ f, (b_2,a_2) $\in \alpha$, (a_2,y$'$) $\in$ g. Now from (x,b_1) $\in$ f and (x,b_2) $\in$ f we get b_1=b_2. Therefore (b_1,a_1) $\in \alpha$, (b_2,a_2) $\in \alpha$ impties that (b_1, a_1) $\in \alpha$, (b_1, a_2) $\in \alpha$. Hence a_1 = a_2. So, (a_1,y) $\in$ g, (a_2, y$'$) $\in$ g implies that y=y$'$. Thus (x,y) $\in$ fαg, (x,y$'$) $\in$ fαg implies that y = y$'$. Next let (x,y) $\in$ fαg and (x$'$,y) $\in$ fαg. Now (x, y) $\in$ fαg implies there exist $a_1 \in$ A, $b_1 \in$ B such that (x,b_1) $\in$ f, (b_1, a_1) $\in \alpha$, (a_1,y) $\in$ g and (x$'$,y) $\in$ fαg implies there exist $a_2 \in$ A, $b_2 \in$ B such that (x$'$,b_2) $\in$ f, (b_2,a_2) $\in \alpha$, (a_2,y) $\in$ g. Then (a_1,y) $\in$ g, (a_2,y) $\in$ g implies that a_1 = a_2.

So, (b_1, a_1) $\in \alpha$, (b_2,a_2) $\in \alpha$ implies that $b_1 = b_2$ and consequently (x,b_1) $\in$ f. (x′,b_2) $\in$ f implies x = x′. Hence (x,y) $\in f\alpha g$, (x′,y) $\in f\alpha g$ implies that x = x′. Therefore $f\alpha g \in$ **J** (A,B). Also, ($f\alpha g$) β h = $f\alpha$ ($g\beta h$). Indeed, (a,b) $\in$ ($f\alpha g$) β h if and only if there exist $a_1 \in$ A, $b_1 \in$ B such that { (a,b_1) $\in f\alpha g$ }, (b_1,a_1) $\in \beta$, (a_1,b) $\in$ h if and only if there exist $a_2 \in$ A, $b_2 \in$ B such that {(a, b_2) $\in$ f, (b_2, a_2) $\in \alpha$, (a_2, b_1) $\in$ g }, (b_1, a_1) $\in \beta$, (a_1 ,b) $\in$ h if and only if (a, b_2) $\in$ f, (b_2, a_2) $\in \alpha$, { (a_2, b_1) $\in$ g, (b_1, a_1) $\in \beta$, (a_1, b) $\in$ h } if and only if (a, b_2) $\in$ f, (b_2, a_2) $\in \alpha$, (a_2, b) $\in g\beta h$ if and only if (a,b) $\in f\alpha$ ($g\beta h$). Thus ($f\alpha g$) $\beta h = f\alpha$ ($g\beta h$). Hence **J** (A, B) is a Γ–semigroup.

Theorem (2. 3) : **J** (A, B) is an inverse Γ-semigroup.

Proof. From Lemma (2. 2) we know that **J** (A, B) is a Γ-semigroup, where Γ=**J** (B,A). First we shall prove that **J** (A, B) is a regular Γ-semigroup.

Let f $\in$ **J** (A, B). Then f : dom f→ran f is one-one. Let f^{-1} : ran f→dom f be such that $ff^{-1} = 1_{\text{dom } f}$ and $f^{-1}f = 1_{\text{ran } f}$, where $1_{\text{dom } f}$ denotes identity mapping on dom f. It is immediate that $f^{-1} \in$ **J** (B,A). Now $f = ff^{-1}ff^{-1}f$ shows that f $\in$ f Γ **J** (A,B) Γ f. Thus **J** (A,B) is regular. Now to prove that J (A, B) is an inverse Γ-semigroup it is sufficient (from THEOREM 2.1) to show that any two α-idempotent of J (A, B) are α commutative where $\alpha \in \Gamma$. So we want to locate α-idempotents of J (A,B).

If an element f $\in$ **J** (A,B) be α-idempotent then $f\alpha = 1_X$, where X = dom f and conversely if $f\alpha = 1_X$ where X = dom f then f is α-idempotent. Now let f, h be any two α-idempotents in J (A B) We shall prove that $f\alpha h = h\alpha f$ ... (1).

Now $f\alpha = 1_X$ where X=dom f and $h\alpha = 1_Y$ where Y = dom h. Then

$$\text{dom } [f\alpha h] = \text{dom}[1_X h] = [X \cap Y] 1_X^{-1} = X \cap Y$$

$$\text{ran } [f\alpha h] = \text{ran } [1_X h] = [X \cap Y] h$$

$\text{dom}[h\alpha f] = \text{dom}[1_Y f] = [Y \cap X]1_Y^{-1} = Y \cap X = X \cap Y$

$\text{ran}[h\alpha f] = \text{ran}[1_Y f] = [Y \cap X]f = [X \cap Y]f$

We shall now show that $[X \cap Y]f = [X \cap Y]h$.

If possible let $[X \cap Y]f \neq [X \cap Y]h$. Then $[X \cap Y]f\alpha \neq [X \cap Y]h\alpha$.

Then $[X \cap Y]1_X \neq [X \cap Y]1_Y$. Then $X \cap Y \neq X \cap Y$, which is absurd. Hence $s\alpha h$ and $h\alpha f$ have same domain and range. Let $a \in X \cap Y$. Then $a = a$ implies $a1_Y = a1_X$ implies $ah\alpha = af\alpha$ implies $ah = af$ (since α is one-one) implies $[a1_X]h = [a1_Y]f$ implies $[af\alpha]h = [ah\alpha]f$ implies $af\alpha h = ah\alpha f$. So $f\alpha h = h\alpha f$. Thus (1) is proved. Hence J (A,B) is an inverse Γ-semigroup.

LEMMA (2.4) : Let M be an inverse Γ-semigroup. If e and f be two idempotents of M such that exe = e, fyf = f, exf = eyf, fye = fxe, then exf = fxe.

Proof. Let $a \in V_u^v(exf)$. Then exfuavexf = exf and avexfua = a. Now fuavexfyuave = fuave. So, fuave is an x-idempodent. Also, exfyfuavexexf = exf and fuavexexfyuave = fuave. Hence $exf \in V_x^y(fuave)$, . . . (1). Again fuavexfuaveyfuave = fuav (exf) uav (eyf) uave = fuav (exf) uav (exf) uave = fuavexfuave = fuave.

Similarly, fuaveyfuavexfuave = fuave. So, $fuave \in V_x^y(fuave)$ . . . (2). From (1) and (2) exf = fuave (since M is an inverse Γ-semigroup). Therefore exf is an x-idempotent of M. Similarly fxe is an x-idempotent of M. Now exfyfxexexf = exfxexf = exf and fxexexfyfxe = fxexfxe = fxe. Hence $fxe \in V_y^x(exf)$, But $exf \in V_y^x(exf)$. So, exf = fxe. This completes the proof.

LEMMA (2.5) : Let M be a regular Γ-semigroup and T be a Γ'-semigroup. Let (f,g) be a homomorphism from (M,Γ) onto (T,Γ'). Let e' be an x'-idempotent of T. Then $e'f^{-1}$ contains an idempotent of M

Proof. Let $a \in M$ be such that $af = e' = e'x'e'$, where $x' \in \Gamma'$. Let $x \in \Gamma$ be such that $xg = x'$. Now let us consider the element axa. As M is a regular Γ—semigroup there exist $c \in M$ and $y, z \in \Gamma$ such that $axayczaxa = axa$ and $czaxayc = c$. Now $aycza$ is an x—idempotent in M since $ayczaxaycza = aycza$

$$\begin{aligned} \text{Also, } (aycza) f &= (af)(yg)(cf)(zg)(af) = e'x'e'(yg)(cf)(zg)\, e'x'e' \\ &= (axaf)(yg)(cf)(zg)(axaf) = (axayczaxa)f = (axa) f \\ &= e'. \end{aligned}$$

Hence $e'f^{-1}$ contains an idempotent of M.

LEMMA (2.6) : Let M be a regular Γ—semigroup and let M′ be a Γ'—semigroup. Let (f, g) be a homomorphism from (M, Γ) onto (M', Γ'). Then M′ is a regular Γ'—semigroup.

Proof. Let $a' = af$ be an element of M′ where $a \in M$. Since M is a regular Γ—semigroup there exist $b \in M$ and $x, y \in \Gamma$ such that $a = axbya$. Then $a' = af = (axbya) f = (af)(xg)(bf)(yg)(af) = a'(xg)(bf)(yg)a'$. Thus a′ is regular. Hence M′ is a regular Γ'—semigroup.

Theorem (2.7) : Let M be an inverse Γ—semigroup and M′ be a Γ'—semigroup. Let (f,g) be a homomorphism from (M, Γ) onto (M', Γ'). Then M′ is an inverse Γ'—semigroup. Moreover, in any homomorphism the (x,y) inverse of an element of M is mapped into the corresponding inverse of the image of the element.

Proof. By LEMMA (2.6) M′ is a regular Γ'—semigroup. We shall show that any two x′—idemotents of M′ are x′—commutative, where $x' \in \Gamma'$. Let e_1', e_2' be two x′—idempotens of M′. Since (f,g) is onto homomorphlsm there exist e_1 and e_2 of M and $x \in \Gamma$ suce that e_1 and e_2 are x—idempotents of M and $e_i f = e'_i$, $xg = x'$, $i = 1,2$ [by LEMMA (2.5)]. As M is an inverse Γ—semigroup $e_1 x e_2 = e_2 x e_1$. Therefore $(e_1 x e_2) f = (e_2 x e_1) f$. So $e_1' x' e_2' = e_2' x' e_1'$. Hence M′ is an inverse Γ'—semigroup. Moreover, for $a \in M$, $a^{-1}_{x,y} f = (af)^{-1}_{xg,\, yg}$. Indeed, $axa^{-1}_{x,y}ya = a$, $a^{-1}_{x,y}yaxa^{-1}_{x,y} = a^{-1}_{x,y}$. Therefore, $(af)(xg)(a^{-1}_{x,y} f)(yg)(af) = af$ and $(a^{-1}_{x,y} f)(yg)(af)(xg)(a^{-1}_{x,y} f) = (a^{-1}_{x,y} f)$. Thus $a^{-1}_{x,y} f = (af)^{-1}_{xg,\, yg}$. This completes the proof.

ACKNOWLEDGEMENT

We express our earnest grattiude to Dr. M. K. Sen for his guidance in the preparation of this paper.

REFERENCE

[1] Sen, M. K. and Saha, N. K. : On Γ—semigroup-I.—to appear in Bull. Cal. Math. Soc,

Department of Mathematics
Pingla Thana Mahabidyalaya
P.O. Maligram, Dist-Midnapore
West Bengal, Pin-721140

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Calcutta-700 019
India

Received—30. 3. 1987