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ON PAIRWISE D-SPACES

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ABSTRACT In this paper, pairwise D-bitopological spaces have been introduced and studied.

KEY WORDS AND PHARASES Pairwise D-Space & D-subset, pairwise semi-open, pairwise semi-continuous, pairwise regularly open, pair-wise D-separation, pairwise D-connected,

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1. **INTRODUCTION** Norman Levine [2] defined and studied D-topologies He called a topological space (X, T) a D-space if T is a D-topology on X, that is, if every nonempty open set of (X, T) is dense in X. It is the aim of this paper to introduce this concept in bitopological spaces. In Section 2, the notion of pairwise D-spaces is introduced and it is shown that a pairwise D-space is not just a pair of D-spaces. Some properties of such pairwise D-spaces have been investigated and certain theorems of [2] have been generalized. In Section 3, we define pairwise D-connected bitopological space and the relation of such concept with that of connected bitopological space is studied.

In course of the exposition of the paper, we shall require some definitions which we state below. By (X, T_1, T_2) or simply by X we shall always mean a bitopological space with two topologies T_1 and T_2, $A^{\circ 1}$, $A^{\circ 2}$ shall mean T_1-interior and T_2-interior respectively of a subset A of X, whereas by $\bar{A}^{T_1}$ and $\bar{A}^{T_2}$ will be denoted respectively the T_1-closure and T_2 closure of A in (X, T_1, T_2,). A bitopological space (X, T_1, T_2) is said to satisfy pairwise T_1-axiom [4] if for any two distinct points x, y of X, there exist T_1-open set U and T_2-open set V such that

$x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. (X, T_1, T_2) is called pairwise Hausdorff [1] if for any pair of distinct points x, y of X, there are T_1-open neighbourhood V of x and T_2-open neighbourhood W of y such that $V \cap W = \Phi$, A subset A of X is called T_i semiopen with respect to T_j [5], if there is a T_i open set V such that $V \subset A \subset \overline{V}^{T_j}$, where i, j=1, 2 and $i \neq j$. A is called pairwise semiopen if it is T_1 semiopen w.r.t. T_2 and T_2 semiopen w.r.t T_1. A function f from (X, T_1, T_2) to a bitopological space (Y, P_1, P_2) will be called T_1 P_1—semi continuous w.r.t. T_2 [5], if for each P_1-open set A, f^{-1} (A) is T_1 semiopen w.r.t. T_2. Similarly T_2 P_2-semi continuity of f w r.t. T_1 is defined. f is called pairwise semi continuous if f is T_1P_1-semicontinuous w.r.t. T_2 and T_2P_2-semi continuous w.r.t T_1. A subset A in (X, T_1, T_2) is called T_1 regularly open w.r.t. T_2 [5], if and only if $A = (\bar{A}^{T_2})^{i_1}$. Similar goes the definition of T_2 regularly open set w.r.t. T_1.

2. PAIRWISE D-SPACES AND THEIR PROPERTIES

DEFINITION 2. 1 Let (X, T_1, T_2) be a bitopological space. We easily note that every nonempty T_1 open set of X is T_2-dense in X if and only if every nonempty T_2 open set is T_1 dense in X.

We define (X, T_1, T_2) to be a pairwise D-space if and only if every nonempty T_1 open set is T_2 dense in X (or equivalently, if and only if every nonempty T_2 open set is T_1 dense in X).

It follows immediately that

THEOREM 2.2 (X. T_1, T_2) is pairwise D-space if and only is every nonempty T_1 open set has nonempty intersection with every nonempty T_2 open set

REMARK 2.3 It follows from definition that a pairwise D-space can never be pairwise Hausdorff, though it may satisfy pairwise T_1-axiom as is seen from

EXAMPLE 2.4 Let X be an uncountable set and T_1 be the cofinite topology and T_2 - the co-countable topology on X. Then (X, T_1, T_2) is a pairwise D-space and satisfies T_1-axiom.

REMARK 2.5 A bitopological space (X, T_1, T_2) is pairwise D-space does not imply that (X, T_1) or (X, T_2) is a D-space. Also, (X, T_1, T_2) may not be a pairwise D-space even if (X, T_1) and (X, T_2) are D-spaces. This is seen in the next example.

EXAMPLE 2.6 Let $X = \{ a, b, c, d \}$, $T_1 = \{ X, \phi, \{ a \} \}$ and $T_2 = \{ X, \phi, \{ b \} \}$. Then (X, T_1) and (X, T_2) are D-spaces but (X, T_1, T_2) is not a pairwise D-space.

Again, let $P_1 = \{ X, \phi, \{ a, b \}, \{ c, d \} \}$ and $P_2 = \{ X, \phi, \{ a, c \}, \{ b, d \} \}$. Then (X, P_i, P_2) is a pairwise D-space whereas neither (X, P_1) nor (X, P_2) is a D-space.

THEOREM 2.7 Every subspace of a pairwise D-space is also a pairwise D-space.

THEOREM 2.8 (X, T_1, T_2) is pairwise D-space if and only if X contains no proper pairwise T_1 regularly open set w.r.t. T_2 as well as no proper T_2 regularly open set w.r.t. T_1.

PROOF Let (X, T_1, T_2) be a pairwise D-space. If A is a proper T_1 regularly open subset of X w. r. t. T_2, then $(\bar{A}^{T_2})^{i}_{1} = A \neq X$ and hence $\bar{A}^{T_2} \neq X$ - a contradiction. Similarly X has no proper T_2 regularly open subset w.r.t. T_1.

We prove a stronger converse that if X has no proper T_1 regularly open set w.r.t. T_2 or no T_2 regularly open set w.r.t. T_1, then (X, T_1, T_2) is a pairwise D-space.

Infact, let (X, T_1, T_2) have no proper T_1 regularly open subset w.r.t. T_2 and let it not be a pairwise D-space. Then there exist nonempty sets $A \in T_1$, $B \in T_2$ such that $A \cap B = \phi$. Then $\bar{A}^{T_2} \cap B = \phi$ and hence $(\bar{A}^{T_2})^{i}_{1} \cap B = \phi$. Then $(\bar{A}^{T_2})^{i}_{1}$ is a proper T_1 regularly open set w.r.t. T_2, which goes against the hypothesis.

Pervin [3] defined a bitopological space (X, T_1, T_2) to be connected if and only if X cannot be expressed as the union of two nonempty disjoint sets A and B such that $(A \cap \bar{B}^{T_2}) \cup (\bar{A}^{T_1} \cap B) = \phi$. He proved that (X, T_1, T_2) is connected if and only if X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is T_1-open and B is T_2-open. Thus we immediately have,

THEOREM 2.9 A pairwise D-space (X, T_1, T_2) is connected.

The next example shows that the converse of the above theorem is false.

EXAMPLE 2.10 Let X = { a, b, c, d }, T_1 = { X, ϕ, { b, c } } and T_2 = { X, ϕ, { b }, { d }, { b, d } }.

Then (X, T_1, T_2) is obviously connected but not a pairwise D-space.

THEOREM : 2.11 Pairwise semi-continuous and hence pairwise continuous image of a pairwise D-space is also so.

COROLLARY : 2.12 Every pairwise semi-continuous function (and hence every pairwise continuous function) from a pairwise D-space into any pairwise Hausdorff space is constant.

THEOREM : 2.13 Let { (X_a, T^1_a, T^2_a) : a ϵ I } be a family of bitopological space (where I is some index set) and (X, T^1, T^2) be their cartesian product, i.e., $X = \pi X_a$ and T^1, T^2 are respectively generated by T^1_a's and T^2_a's. (X, T^1, T^2) is a pairwise D-space if and only if each (X_a, T_a^1, T_a^2) is a pairwise D-space.

PROOF. Since the projection map P_a : (X, T^i) $\rightarrow$ (X, T_a^i) is continuous and open for each a ϵ I and i = 1, 2, by virtue of Theorem 2 11 the necessity follows.

Conversely, suppose (X_a, T^1_a, T^2_a) is a pairwise D-space, for each a ϵ I and V be a nonempty T^1-open set in X and x ϵ V. Then there exist indices $a_1, a_2, \ldots, a_k$ (say) and $U_{a_i} \epsilon T^1_{a_i}$ such that

$$x \epsilon \bigcap_{i=1}^{k} P_{a_i}^{-1} (U_{a_i}) \subset V.$$

Then $X = \bigcap_{i=1}^{k} \left\{ P_{a_i}^{-1} (X_{a_i}) \right\} = \bigcap_{i=1}^{k} \left\{ P_{a_i}^{-1} (\overline{U_{a_i}}^{T^2_{a_i}}) \right\}$

$$\overline{P_{\alpha_1}^{-1}(U_{\alpha_1})}^{T_\beta} = \overline{P_{\alpha_1}^{-1}(U_{\alpha_1})}^{T_\beta} \subset \overline{V}^{T_\beta} \subset X. \quad \text{Thus } \overline{V}^{T_\beta} \neq X.$$

THEOREM 2.14 (a) Let $f : (X, T_1, T_2) \to (Y, P_1, P_2)$ be one-to-one and pairwise open and (Y, P_1, P_2) is a pairwise D-space, then (X, T_1, T_2) is also so.

(b) Let $f : (X, T_1, T_2) \to (Y, P_1, P_2)$ be surjective and $\{ f^{-1}(V) : V \in P_i \} = T_i$, for $i = 1, 2$, then $(X, T_1, T_2,)$ is a pairwise D-space if and only if (Y, P_1, P_2) is so.

REMARK 2.15 The conclusion of Theorem 2.14 (a) may not be true in case either $f : (X, T_1) \to (Y, P_1)$ or $f : (X, T_2) \to (Y, P_2)$ is open but not both. This is seen from

EXAMPLE 2.16 Let $X = Y = \{a, b, c, d\}$,

$T_1 = \{ X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, c, b\} \}$, $T_2 = \{ \phi, X, \{b\} \}$,

$P_1 = \{ X, \phi, \{b\}, \{a, b\} \}$ and $P_2 = \{ X, \phi, \{b\}, \{b, d\} \}$.

Consider the identity map $f : (X, T_1, T_2) \to (Y, P_1, P_2)$. Then $f : (X, T_2) \to (Y, P_2)$ is open but $f : (X, T_1) \to (Y, P_1)$ is not open. (Y, P_1, P_2) is a pairwise D-space, whereas (X, T_1, T_2) is not so.

DEFINITION 2.17 A subset A of (X, T_1, T_2) is called a pairwise D-subset of X if and only if $(A, (T_1)^A, (T_2)_A)$ is a pairwise D-space.

THEOREM 2.18 Let $A \subset Y \subset (X, T_1, T_2)$. Then A is a pairwise D-subset of X if and only if it is a pairwise D-subset of $(Y, (T_1)_Y, (T_2)^Y)$.

PROOF. Easy and left.

THEOREM 2.19 If A is a pairwise D-subset of (X, T_1, T_2), then $\bar{A}^{T_1} \cap \bar{A}^{T_2}$ is also a pairwise D-subset of X

PROOF. If $B = \bar{A}_{T_1} \cap \bar{A}_{T_2}$ is not a pairwise D-subset, then there will exist two nonempty disjoint sets U and V such that $U \in (T_1)_B$ and $V \in (T_2)_B$. Then $U \cap A$ ($\neq \phi$) and $V \cap A$ ($\neq \phi$) are respectively $(T_1)_A$ and $(T_2)_A$ open such that $(U \cap A) \cap (V \cap A) = \phi$ which proves that A is not a pairwise D-space.

REMARK 2.20 If A is a pairwise D-subset of (X, T_1, T_2) then neither $\bar{A}_{T_1}$ nor $\bar{A}_{T_2}$ may be pairwise D-subset as is seen from

EXAMPLE 2.21 Let $X = \{a, b, c, d, f\}$,

$T_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and

$T_2 = \{\phi, X, \{a\}, \{f\}, \{a, f\}, \{a, c\}, \{a, c, f\}\}$

and consider the set $A = \{a, c\}$ of the bitopological space (X, T_1, T_2).

Now, $(T_1)_A = \{A, \phi, \{a\}\}$ and $(T_2)_A = \{A, \phi, \{a\}\}$. A is obviously a pairwise D-subset of X.

Now, $\bar{A}_{T_1} = \{a, c, d, f\} = B$ (say) and $\bar{A}_{T_2} = \{a, b, c, d\} = D$ (say) $(T_1)_B = \{B, \phi, \{a\}, \{a, c\}\}$ and $(T_2)_B = \{B, \phi, \{a\}, \{f\}, \{a, f\}, \{a, c\}, \{a, c, f\}\}$; $(T_1)_D = \{D, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $(T_2)_D = \{D, \phi, \{a\}, \{a, c\}\}$.

Clearly neither B nor D is a pairwise D-subset of X.

REMARK 2.22 Union of two pairwise D-subsets of a bitopological space may not be a pairwise D-subset as is seen form

EXAMPLE 2.23 Consider (X, T_1, T_2) of Example 2.21. Let $A = \{a, c\}$ and $B = \{b, d\}$ It has been shown in Example 2.21 that A is a pairwise D-subset. Now, $(T_1)_B = \{B, \phi, \{b\}\}$ and $(T_2)_B = \{\phi, B\}$. Clearly B is also a pairwise D-subset of X. But $A \cup B = \{a, b, c, d\}$ is not a pairwise D-subset as shown in Example 2.21.

THEOREM 2.24 If A and B are pairwise D-subsets of (X, T_1, T_2) and if either $(A \cap B'_2) \neq \phi$ and $(B \cap A'_2) \neq \phi$, or $A \cap B'_1 \neq \phi$ and $A'_1 \cap B \neq \phi$, then $A \cup B$ is a pairwise D-subset of X.

PROOF. Denying the theorem, we get a pair of nonempty sets U and V in X such that $U \in T_1$, $V \in T_2$, $(U \cap V) \cap (A \cup B) = \phi$, but $U \cap (A \cup B) \neq \phi$ and $V \cap (A \cup B) \neq \phi$. Since A and B are pairwise D-subsets, U and V cannot both intersect A or both intersect B. Suppose $U \cap A = \phi$. Since $U \cap (A \cup B) \neq \phi$, we have $U \cap B \neq \phi$ and then $V \cap B = \phi$ and $V \cap A \neq \phi$. If the first assumption of the theorm holds then $U \cap B$ and $A^{i}_{2} \cap B$ are nonempty $(T_1)_B$ and $(T_2)_B$ open subsets of B such that $(U \cap B) \cap (A^{i}_{2} \cap B) \subset U \cap A = \phi$, proving that B is not a pairwise D-subset. If the second assumption of the theorem holds, then $B^{i}_{1} \cap A$ and $V \cap A$ are nonempty and respectively $(T_1)_A$ and $(T_2)_A$ open subsets such that $(B^{i}_{1} \cap A) \cap (V \cap A) \subset V \cap B = \phi$, proving that A is not a pairwise D-subset. The case when $V \cap A = \phi$ can similarly be tackled.

THEOREM 2.25 Union of a chain of pairwise D-subsets of a bitopological space is also a pairwise D-subset.

THEOREM 2.26 Every pairwise D-subset A of a bitopological space is contained in a maximal pairwise D-subsets of the space.

PROOF. Follows from Theorem 2 25 and Zorn's lemma.

COROLLARY 2.27 Every bitopological space is the union of its maximal pairwise D-subsets.

REMARK 2.28 In a bitopological space, maximal pairwise D-subspaces need not be either T_1-open (or T_1-closed) or T_2 open (or T_2-closed). Also maximal pairwise D-subsets need not be disjoint. These are seen from

EXAMPLE 2 29 Consider the bitopological space (X, T_1, T_2) of Example 2.21. It can be checked that the maximal pairw se D-subset containing $A = \{a, c\}$ is $M = \{a. c, d\}$. But M is neither T_1-open nor T_1-closed nor T_2-open nor T_2-closed.

Again the subset $B = \{c, d\}$ is a pairwise D-subset. If B^* denoted the maximal pairwise D-subset of X containing B, then $B^* \subset M$ and hence $B^* \cap M \neq \phi$.

3. PAIRWISE D-CONNECTEDNESS

DEFINITION 3.1 Let (X, T_1, T_2) be a bitopological space. Two nonempty subsets A, B are said to form a pairwise D-seperation of X (where A is T_1-closed and B is T_2-closed) if and only if $X = A \cup B$ and $(A^i_2 \cap B) \cup (A \cap B^i_1) = \phi$. (X, T_1, T_2) is said to be pairwise D-connected if and only if there exists no pairwise D-separation of X.

THEOREM 3.2 A pairwise D-connected space (X, T_1, T_2) is connected but the converse is false.

PROOF. Let X be not connected. Then there exist nonempty T_1-open set A and T_2-open set B such that $X = A \cup B$ and $A \cap B = \phi$. Then $C = X\text{-}A = B$ and $D = X\text{-}B = A$ are nonempty sets, respectively T_1 closed and T_2-closed and are such that $X = C \cup D$ and $(C^i_2 \cap D) \cup (C \cap D^i{}_1) = (C \cap D) \cup (C \cap D) = B \cap A = \phi$. Hence C and D form a pairwise D-separation of X and the first part of the theorem is proved.

To show that the converse is indeed false, we consider the following example.

Let $X = \{ a, b, c, d \}$, $T_1 = \{ X, \phi, \{ a \}, \{ a, b \}, \{ a, d \}, \{ a, b, d \} \}$ and $T_2 = \{ X, \phi, \{ a \}, \{ b \}, \{ a, b \} \}$. Then (X, T_1, T_2) is connected.

Now, let $A = \{ b, c \}$ and $B = \{ a, c, d \}$. Then A, B are respectively T_1-closed and T_2-closed such that $(A^i_2 \cap B) = \{ b \} \cap B = \phi$, $A \cap B^i_1 = \{ b, c \} \cap \{ a, d \} = \phi$ and $X = A \cup B$. Hence A, B form a pairwise D-separation of X and then X is not pairwise D-connected.

THEOREM 3.3 If A, B form a pairwise D-separation of (X, T_1, T_2) and Y is a pairwise D-subset of X, then either $Y \subset A$ or $Y \subset B$.

PROOF. If $Y \not\subset A$ and $Y \not\subset B$, then $Y \cap (X\text{-}A)$ and $Y \cap (X\text{-}B)$ are nonempty disjoint sets in Y, respectively $(T_1)_Y$ open and $(T_2)_Y$ open. Also $[Y \cap (X\text{-}A)] \cap [Y \cap (X\text{-}B)] \subset (X\text{-}A) \cap (X\text{-}B) = X\text{-}A \cup B = X\text{-}X = \phi$. Then Y is not a pairwise D-subset of X. Hence either $Y \subset A$ or $Y \subset B$.

LEMMA 3.4 (X, T_1, T_2) is pairwise D-space if and only if there doesnot exist any pair of nonempty sets A, B respectively T_1-closed and T_2-closed such that X = A $\cup$ B,

THEOREM 3.5 (X, T_1, T_2) is a pairwise D-space if and only if X is pairwise D-connected.

PROOF. Necessity follows from Lemma 3.4.

Conversely, let (X, T_1, T_2) be not a pairwise D-space. Then by Theorem 2 8, there exists a nonempty proper T_1 regularly open set A w.r.t. T_2. Then $(\bar{A}^{T_2})^i_1$ = A. Let G = X-A and H = $\bar{A}^{T_2}$ G and H are nonempty sets respectively T_1-closed and T_2-closed. Also, $G^i_2 \cap H = (X-\bar{A}^{T_2}) \cap \bar{A}^{T_2} = \phi$ and $G \cap H^i_1 = (X-A) \cap A = \phi$.

Then G and H form a pairwise D-separation of X, proving that X is not pairwise D-connected.

REMARK 3.6 It is shown in [3] that a space (X, T_1, T_2) is connected iff every pairwise continuous map of X into the bitopological 2-space (Y, D_1, D_2) is constant, where Y = { 0, 1 }, D_1 = { Y, ϕ, { 0 } } and D_2 = { Y, ϕ. { 1 } }. Now by Theorem 3.5 and 2.11 we can easily see that if X is pairwise D-connected, then even every pairwise semi-continuous map of X into the bitopological 2 space is constant. But by taking X = [a, b, c, d], T_1 = { X, ϕ, { a }, { a, b }, { a, d }, { a, b, d } } and T_2 = { X, ϕ, { a }, { b }, { a, b } } (see converse part of Theorem 3 2) we observe that every pairwise continuous function from (X, T_1, T_2) into the bitopological 2-space is constant, although X is not pairwise D-connected.

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