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A SYSTEM OF AFFINE CONNECTIONS ON A RIEMANNIAN MANIFOLD

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O. In 1950, Sen [4] constructed an algebraic system generated by a single element. In course of his investigation he obtained a cyclic sequence under certain conditions. Later Chaki [1] started with the coefficients of an arbitrary affine connection in a Riemannian space and constructed a cyclic sequence of twelve distinct affine connections. In 1961 Gupta [2] obtained several interesting results regarding the sequence obtained by Chaki.

In section one of the present paper, the sequence of affine connections constructed by Chaki has been generalized on a Riemannian manifold and the corresponding sequence of torsion tensors has been constructed. A few interesting results regarding this sequence have been obtained. Section two deals with the covariant derivatives of the Riemann metric with respect to the affine connections of the sequence. Finally, in section three, the curvature tensors of the connections of the sequence have been calculated.

1. Let M be a Riemannian manifold with Riemann metric g and let ∇ be an arbitrary affine connection defined on M. Write

$$(1.1) \qquad a = g(\nabla_Y Z, X)$$

where X, Y, Z, ... are differentiable vectorfields on M. The associate a^* and the conjugate a' of a are defined by

$$(1.2) \qquad \begin{cases} a^* = g(\nabla_Y Z, X) + (\nabla_Y g)(Z, X) \\ a' = g(\nabla_Z Y, X) \end{cases}$$

Evidently, $a^{**} = a'' = a$. The connection a is said to be self-associate or self-conjugate according as $a^* = a$ or $a' = a$ respectively.

Let H (Y, Z) be a differentiable tensorfield of type (1.2) on M. Define an affine connection $\overset{1}{\nabla}$ on M by the relation

$$(1.3) \quad \overset{1}{\nabla}_Y Z = \nabla_Y Z + H(Y, Z)$$

and write

$$(1.4) \quad d_1 = g(\overset{1}{\nabla}_Y Z, X)$$

Starting with d_1 a sequence of affine connections can be obtained by forming successively the associate and conjugate of d_1, and the elements of this sequence may be denoted by d_1, d_2, d_3, ... etc.

Thus

$$d_2 = d_1^* = g(\overset{2}{\nabla}_Y Z, X) = g(\overset{1}{\nabla}_Y Z, X) + (\overset{1}{\nabla}_Y g)(Z, X)$$

$$d_3 = d_1^{*\prime} = d'_2 = g(\overset{3}{\nabla}_Y Z, X) = g(\overset{2}{\nabla}_Z Y, X)$$

and so on. Proceeding in this way a cyclic sequence of twelve affine connections can be obtained which is as follows :

$$(I) \quad \left| \begin{aligned} d_1 &= a + \gamma \\ d_2 &= a + \alpha - \delta \\ d_3 &= a' + \alpha' - \delta' \\ d_4 &= a + \alpha - \lambda + \beta + \varepsilon' \\ d_5 &= a' + \alpha' - \lambda + \beta' + \varepsilon \\ d_6 &= a + \alpha + \alpha' - \lambda + \beta + \beta' - \gamma' \\ d_7 &= a' + \alpha + \alpha' - \lambda + \beta + \beta' - \gamma \\ d_8 &= a' + \alpha' - \lambda + \beta + \beta' + \delta \\ d_9 &= a + \alpha - \lambda + \beta + \beta' + \delta' \\ d_{10} &= a' + \alpha' + \beta' - \varepsilon' \\ d_{11} &= a + \alpha + \beta - \varepsilon \\ d_{12} &= a' + \gamma' \end{aligned} \right.$$

where

$$
(\mathrm{II})\quad
\left|
\begin{array}{ll}
\alpha = (\nabla_Y g)(Z, X), & \alpha' = (\nabla_Z g)(Y, X) \\
\lambda = (\nabla_X g)(Y, Z) = \lambda' & \\
\beta = g(\nabla_Y X - \nabla_X Y, Z), & \beta' = g(\nabla_Z X - \nabla_X Z, Y) \\
\gamma = g(H(Y, Z), X), & \gamma' = g(H(Z, Y), X) \\
\delta = g(H(Y, X), Z), & \delta' = g(H(Z, X), Y) \\
\epsilon = g(H(X, Y), Z\ , & \epsilon' = g(H(X, Z), Y)
\end{array}
\right.
$$

It is interesting to see that the affine connection 'd' defined by

$$(1.5)\quad 2d = d_k + d_{k+6},\ k = 1, \dots, 6$$

is both self-associate and self-conjugate, that is,

$$d^* = d \text{ and } d' = d$$

and it is given by

$$(1.6)\quad 2d = a + a' + \alpha + \alpha' - \lambda + \beta + \beta'$$

If $\overset{k}{T}(Y, Z) = \overset{k}{\nabla}_Y Z - \overset{k}{\nabla}_Z Y - [Y, Z],\ k = 1, 2, \dots, 12,$

denotes the torsion tensor of the affine connection $\overset{k}{\nabla}$, where $d_k = g(\overset{k}{\nabla}_Y Z, X)$, then a straightforward calculation shows that

$$
(1.7)\quad
\left|
\begin{array}{l}
\overset{2}{T}(Y, Z) = \overset{9}{T}(Y, Z) = -\overset{3}{T}(Y, Z) - 2[Y, Z] = -\overset{8}{T}(Y, Z) - 2[Y, Z] \\
\overset{4}{T}(Y, Z) = \overset{11}{T}(Y, Z) = -\overset{5}{T}(Y, Z) - 2[Y, Z] = -\overset{10}{T}(Y, Z) - 2[Y, Z] \\
\overset{6}{T}(Y, Z) = \overset{1}{T}(Y, Z) = -\overset{7}{T}(Y, Z) - 2[Y, Z] = -\overset{12}{T}(Y, Z) - 2[Y, Z]
\end{array}
\right.
$$

Theorem (1.1) : The torsion tensors of the sequence (I) satisfy

$$(1.8)\quad \overset{k}{T}(Y, Z) + \overset{k+9}{T}(Y, Z) = -2[Y, Z]$$

The proof is immediate.

It is easy to see that the torsion tensor $\tau(Y, Z)$ of the connection 'd' defined in (1.5) is

$$(1.9) \quad \tau(Y,Z) = \overset{k}{T}(Y, Z) + \overset{k+6}{T}(Y, Z) = -2[Y, Z]$$

It is known that every Riemannian manifold admits a Riemann connection or Levi-Civita connection which is a metric connection and whose torsion tensor is zero. If $\overline{\nabla}$ denotes this L. C. connection, then [3].

$$2g(\overline{\nabla}_Y Z, X) = Yg(Z, X) + Zg(Y,X) - Xg(Y, Z)$$
$$+ g([X, Y], Z) + g([X, Z], Y) + g([Y, Z], X)$$

Writing $d = g(D_Y Z, X)$ it can be seen, by virtue of (1.9),

$$(1.10) \quad 2g(\overline{\nabla}_Y Z, X) = 2g(D_Y Z, X) - g(\tau(X, Y), Z) - g(\tau(X, Z), Y)$$
$$- g(\tau(Y, Z), X)$$

Theorem (1.2) : In terms of the connections of the sequence (I), the L.C. connection of the Riemann manifold is given by (1.10).

2. In this section symmetric covariant derivatives of the Riemann metric g have been dealt with.

Covariant derivative of the Riemann metric g is said to be symmetric with respect to an affine connection ∇ if

$$(2.1) \quad (\nabla_Y g)(Z, X) = (\nabla_Z g)(Y, X)$$

for all differentiable vectorfields on M. In this section the above relation will be considered to be true.

If $(\overset{k}{\nabla}_Y g)(Z, X) = Yg(Z, X) - g(\overset{k}{\nabla}_Y Z, X) - g(Z, \overset{k}{\nabla}_Y X)$, $k = 1, 2, \ldots, 12$,

denote the covariant derivative of the Riemann metric g with respect to the affine connection $\overset{k}{\nabla}$, then

$$(2.2)\quad \left| \begin{array}{l} (\overset{1}{\nabla}_Y g)(Z, X) = -(\overset{2}{\nabla}_Y g)(Z, X) = -(\overset{7}{\nabla}_Y g)(Z, X) = \\ (\overset{8}{\nabla}_Y g)(Z, X) \\ \\ (\overset{3}{\nabla}_Y g)(Z, X) = -(\overset{4}{\nabla}_Y g)(Z, X) = -(\overset{9}{\nabla}_Y g)(Z, X) = \\ (\overset{10}{\nabla}_Y g)(Z, X) \\ \\ (\overset{5}{\nabla}_Y g)(Z, X) = -(\overset{6}{\nabla}_Y g)(Z, X) = -(\overset{11}{\nabla}_Y g)(Z, X) = \\ (\overset{12}{\nabla}_Y g)(Z, X) \end{array} \right.$$

From (2.2), (2.1) and (II) it can be seen that

(i) $\overset{1}{\nabla} g, \overset{2}{\nabla} g, \overset{7}{\nabla} g, \overset{8}{\nabla} g$ are symmetric if and only if
$\delta + \gamma - \varepsilon - \varepsilon' = O$;

(ii) $\overset{3}{\nabla} g, \overset{4}{\nabla} g, \overset{9}{\nabla} g, \overset{10}{\nabla} g$ are symmetric if and only if
$2\beta + \beta' + \tau - \gamma + \varepsilon' - \delta' + \gamma' = O$;

(iii) $\overset{5}{\nabla} g. \overset{6}{\nabla} g, \overset{11}{\nabla} g, \overset{12}{\nabla} g$ are symmetric if and only if
$2\beta + \beta' + \tau - \varepsilon + \delta + \delta' - \gamma' = O$

Theorem (2.1) : In view of the symmetry of the covariant derivatives of the Riemann metric g, the elements of the sequence (I) can be divided into three classes $\{ d_1, d_2, d_7, d_8 \}$, $\{ d_3, d_4, d_9, d_{10} \}$ and $\{ d_5, d_6, d_{11}, d_{12} \}$.

Theorem (2.2) : If the covariant derivative of g is symmetric with respect to the connections in any two of the above classes, then it is symmetric with respect to the connections of the remaining class also.

3. The curvature tensor K (X, Y) Z of the affine connection ∇ is given by

$$(3.1)\qquad K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and its fully covariant form 'K (X, Y, Z, W) is given by

$$(3.2)\qquad {}'K(X,Y,Z,W) = g(K(X,Y)Z,W)$$

If $\overset{\iota}{{}'K}(X,Y,Z,W)$, $\iota = 1, 2, \ldots, 12$ denote the fully covariant curvature tensors of the affine connections of the sequence (I , then they form the following sequence :

(III)

$$\overset{1}{{}'K}(X,Y,Z,W) = {}'K(X,Y,Z,W) + g((\nabla_X H)(Y,Z),W) - g((\nabla_Y H)(X,Z),W) + g(H(T(X,Y),Z),W) + g(H(X,H(Y,Z)),W) - g(H(Y,H(X,Z)),W)$$

$$\overset{2}{{}'K}(X,Y,Z,W) = -\overset{1}{{}'K}(X,Y,W,Z)$$

$$2\,\overset{3}{{}'K}(X,Y,Z,W) = 2\,\overset{2}{{}'K}(X,Y,Z.W) + g((\overset{3}{\nabla}_Y \overset{3}{T})(X,Z),W) - g((\overset{3}{\nabla}_X \overset{3}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{3}{T}(\overset{3}{T}(X,Y),Z),W)$$

$$\overset{4}{{}'K}(X,Y,Z,W) = -\overset{3}{{}'K}(X,Y,W,Z)$$

$$2\,\overset{5}{{}'K}(X,Y,Z,W) = 2\,\overset{4}{{}'K}(X,Y,Z,W) + g((\overset{5}{\nabla}_Y \overset{5}{T})(X,Z),W) - g((\overset{5}{\nabla}_X \overset{5}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{5}{T}(\overset{5}{T}(X,Y),Z),W)$$

$$\overset{6}{{}'K}(X,Y,Z,W) = -\overset{5}{{}'K}(X,Y,W,Z)$$

$$2\,\overset{7}{{}'K}(X,Y,Z,W) = 2\,\overset{6}{{}'K}(X,Y,Z,W) + g((\overset{7}{\nabla}_Y \overset{7}{T})(X,Z),W) - g((\overset{7}{\nabla}_X \overset{7}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{7}{T}(\overset{7}{T}(X,Y),Z),W)$$

$$\overset{8}{{}'K}(X,Y,Z,W) = -\overset{7}{{}'K}(X,Y,W,Z)$$

$$\left|\begin{aligned}
&2\,\overset{9}{'K}(X,Y,Z,W) = 2\,\overset{8}{'K}(X,Y,Z,W) + g((\overset{9}{\nabla}_Y \overset{9}{T})(X,Z),W) \\
&\qquad - g((\overset{9}{\nabla}_X \overset{9}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{9}{T}(\overset{9}{T}(X,Y),Z),W) \\
&\overset{10}{'K}(X,Y,Z,W) = -\,\overset{11}{'K}(X,Y,W,Z) \\
&2\,\overset{11}{'K}(X,Y,Z,W) = 2\,\overset{10}{'K}(X,Y,Z,W) + g((\overset{11}{\nabla}_Y \overset{11}{T})(X,Z),W) \\
&\qquad - g((\overset{11}{\nabla}_X \overset{11}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{11}{T}(\overset{11}{T}(X,Y),Z),W) \\
&\overset{12}{'K}(X,Y,Z,W) = -\,\overset{11}{'K}(X,Y,W,Z)
\end{aligned}\right.$$

Assuming that

$$(3.3)\quad \left|\begin{aligned}
&\overset{1}{T}(Y,Z) = \overset{2}{T}(Y,Z) = \overset{4}{T}(Y,Z) = \overset{6}{T}(Y,Z) = \overset{9}{T}(Y,Z) = \overset{11}{T}(Y,Z) = O \\
&\text{and consequently,} \\
&\overset{3}{T}(Y,Z) = \overset{5}{T}(Y,Z) = \overset{7}{T}(Y,Z) = \overset{8}{T}(Y,Z) = \overset{10}{T}(Y,Z) = \overset{12}{T}(Y,Z) = \\
&-2[Y,Z]
\end{aligned}\right.$$

from sequence (**III**) we find

(i) $\overset{l}{'K}(X,Y,W,Z) + \overset{l+1}{'K}(X,Y,Z,W) = O$ for $l = 1, 3, 5, 7, 9, 11,$

(ii) $$2\,\{\overset{l+1}{'K}(X,Y,Z,W) - \overset{l}{'K}(X,Y,Z,W)\} = g((\overset{l+1}{\nabla}_Y \overset{l+1}{T})(X,Z),W) - g((\overset{l+1}{\nabla}_X \overset{l+1}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{l+1}{T}(\overset{l+1}{T}(X,Y),Z),W)$$

for $l = 2, 4, 6,$

(iii) $2\,\{\overset{l}{'K}(X,Y,Z,W) - \overset{l+1}{'K}(X,Y,Z,W) = g((\overset{l}{\nabla}_Y \overset{l}{T})(X,Z),W)$

$$- g((\overset{l}{\nabla}_X \overset{l}{T})(Y, Z), W) \quad - \tfrac{1}{2} g(\overset{l}{T}(\overset{l}{T}(X, Y), Z, W$$

for $l = 8, 10, 12.$

Writting

$t(Y, Z) = 2[Y, Z]$, the common value of $\overset{3}{T}, \overset{5}{T}, \overset{7}{T}, \overset{8}{T}, \overset{10}{T}, \overset{12}{T}$ and

$\overset{1}{D} = \overset{3}{\nabla} + \overset{7}{\nabla} - \overset{10}{\nabla}$, $\overset{2}{D} = \overset{5}{\nabla} - \overset{8}{\nabla} - \overset{12}{\nabla}$

we find from sequence (III), that

$$(3.4) \quad g((\overset{1}{D}_Y t)(X, Z), W) - g((\overset{1}{D}_X t)(Y, Z), W) - \tfrac{3}{2} g(t(t(X, Y), W)$$

$$= g((\overset{2}{D}_Y t)(X, W), Z) - g((\overset{2}{D}_X t)(Y, W), Z) - \tfrac{3}{2} g(t(t(X, Y), W), Z)$$

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