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SOME RESULTS ON FIXED POINT THEOREM IN COMPACT METRIC SPACE.

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§ 1 : INTRODUCTION :—

Let (X, d) metric space and $T : X \to X$ be an operator. T is said to be a contraction mapping if it satisfies

1) $d(Tx, Ty) \leqslant qd(x, y), \forall\, x, y \in X, 0 < q < 1.$

and T is said to be a contractive mapping if it satisfies

2) $d(Tx, Ty) < qd(x, y), \forall\, x, y \in X, q=1.$

A contraction mapping is contractive but the converse is clearly not true.

If (X, d) be a complete metric space and $T : X \to X$ satisfies (1) then by Banach's fixed point theorem, T has a unique fixed point but if $T : X \to X$ satisfy (2) then T need not have a fixed point.

Again if (X, d) is a compact matric space and $T : X \to X$ satisfies (2) then T has a unique fixed point. Moreover, it follows from this fact that on a compact metric space the notions contraction and of contractive mappings coincide.

On compact metric space, many authors proved fixed point theorems using various contractive type of mappings. Of them, B. Fisher [3] proved the following theorem using contractive type mapping yielding a unique fixed point.

THEOREM – A : (Fisher)

If T is a continuous mapping of a compact metric sapce (X, d) into itself such that

$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)]$

for all x, $y \in X$ with $x \neq y$, then T has a unique fixed point.

S. K. Chatterjea [1] proved the following theorem involving G. Hardy and T. Rogers [4] type contractive mapping yielding a unique fixed point.

THEOREM ; B : (Chatterjea) -

If T is a continuous mapping of a compact metric space (X, d) into itself such that $d(Tx, Ty) < a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y\ Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$ for all distinct $x, y \in X$, $a_i \geqslant \Phi$ ($i = 1, 2, \ldots, 5$) and $\sum_1^5 a_i = 1$, then T has a unique fixed point.

In fact, the theorem of Fisher is a particular case of the theorem due to Chatterjea.

In this paper we have tried to make extension of the results of Fisher and Chatteriea.

δ 2 : FIXED POINT THEOREMS :—

In this section we have proved the following theorems.

THEOREM-1 :

Let (X, d) be a compact metric space and T_1, T_2 be a pair of continuous self mappings defined on X for which there exists non-negative real numbers q_i ($j=1, 2,\ldots,5$) with $\sum_{j=1}^{5} q_j = 1$ such that $d(T_1 x, T_2 y) < q_1 d(x, y) + q_2 d(x, Tx) + q_3 d(y, Ty) + q_4 d(x, Ty) + q_5 d(y, Tx)$ for all distinct $x, y \in X$, then T_1 and T_2 have a unique common fixed point in X.

PROOF : We first define a function $F : X \rightarrow R_+$ as follows :

$F(X) = d(x, T_1 x)$, $\forall\ x \in X$. Clearly F is continuous.

Let $F(z) = \inf\{F(x) : x \in X\}$, which must exist since X is compact.

Now, if possible, suppose that $z \neq T_1 z$ and $T_2 T_1 z \neq T_1 z$. Then $F(T_2 T_1 z) = d(T_2 T_1 z, T_1 T_2 T_1 z) = d(T_1 T_2 T_1 z, T_2 T_1 z)$

$< q_1 d(T_2 T_1 z, T_1 z) + q_2 d(T_2 T_1 z, T_1 T_2 T_1 z) + q_3 d(T_1 z, T_2 T_1 z)$
$+ q_4 d(T_2 T_1 z, T_2 T_1 z) + q_5 d(T_1 z, T_1 T_2 T_1 z)$

$\Rightarrow (1 - q_2 - q_5) F(T_2 T_1 z) < (1 - q_2 - q_4) d(T_1 z, T_2 T_1 z)$ ......(i)

Again, by symmetry, we have

$(1 - q_3 - q_4) F(T_2 T_1 z) < (1 - q_3 - q_5) d(T_1 z, T_2 T_1 z)$ ..... (ii)

Adding (i) and (ii) we get

$$\{2-(q_2 + q_3 + q_4 + q_5)\} F(T_2 T_1 z) < \{2-(q_2 + q_3 + q_4 + q_5)\} d(T_1 z, T_2 T_1 z)$$

$\Rightarrow F(T_2 T_1 z) < d(T_1 z, T_2 T_1 z)$ .... (I)

Now, $d(T_1 z, T_2 T_1 z) < q_1 d(z, T_1 z) + q_2 d(z, T_1 z) + q_3 d(T_1 z, T_2 T_1 z) + q_4 d(z, T_2 T_1 z) + q_5 d(T_1 z, T_1 z)$

$\Rightarrow (1-q_3-q_4) d(T_1 z, T_2 T_1 z) < (1-q_3-q_5) F(z).$

By symmetry, we have

$(1-q_2-q_5) d(T_1 z, T_2 T_1 z < (1-q_2-q_4) F(z).$

Adding these two we get

$$\{2-(q_2 + q_3 + q_4 + q_5)\} d(T_1 z, T_2 T_1 z) < \{2-(q_2 + q_3 + q_4 + q_5)\} F(z)$$

$\Rightarrow \quad d(T_1 z, T_2 T_1 z) < F(z)$ .... (II)

From (I) and (II) we get $F(T_2 T_1 z) < F(z)$ — which is a contradiction.

Hence $T_2 T_1 z = T_1 z$ and $T_1 z = z$.

Therefore, $T_2 z = z$.

Consequently, z is a common fixed point of T_1 and T_2 both.

We shall now show the unicity of the fixed point z.

Suppose that $u (\neq z) \in X$ is another common fixed point of T_1 and T_2.

Then, $d(z, u) = d(T_1 z, T_2 u)$

$< q_1 d(z, u) + q_2 d(z, T_1 z) + q_3 d(u, T_1 u) + q_4 d(z, T_2 u) + q_5 d(u, T_1 z)$

$= (q_1 + q_4 + q_5) d(z, u) = (1-q_2-q_3) d(z, u)$

$\Rightarrow \quad (q_2 + q_3) d(z, u) < 0$ — a contradiction.

So, $d(z, u) = 0 \Rightarrow z = u$.

Consequently, z is the unique common fixed point of T_1 and T_2.

This completes the proof of the theorem.

NOTE :

I) If we take $T_1 = T_2 = T$ and $q_1 = q_4 = q_5 = O$ and $q_2 = q_3 = \frac{1}{2}$ then we get theorem-A.

II) If we take $T_1 = T_2 = T$ we get theorem-B.

THEOREM - 2 : If T be a mapping of a compact metric space (X, d) into itself such that for some fixed positive integer m, T^m satisfies the inequality.

$$d(T^m x, T^m y) < q_1 d(x, y) + q_2 d(x, T^m x) + q_3 d(y, T^m y) + q_4 d(x, T^m y) + q_5 d(y, T^m x)$$

for all $x, y \in X$, $x \neq y$ where $q_i \geqslant O$ with $\sum_{i=1}^{5} q_i = 1$ and if T^m is continuous then T has a unique fixed point in X.

PROOF : In view of Theorem-B, T^m has a unique fixed point z in X. Now, since $T^m z = z$, we have

$Tz = T(T^m z) = T^m(Tz) = Tz \Rightarrow Tz$ is a fixed point of T^m in X, but T^m has a unique fixed point z in X.

Hence, $Tz = z$.

Therefore, z, being unique, is a unique fixed point of T in X.

NOTE :

We can show, by an example, that Theorem-2 is stronger than Theorem-B.

EXAMPLE :- We take X=R and we define $T : R \rightarrow R$ as follows ;-

$$Tx = 0 \text{ if } 0 \leqslant x < \tfrac{1}{2}$$
$$= \tfrac{1}{2} \text{ if } \tfrac{1}{2} \leqslant x \leqslant 1.$$

So, $T^2 x = 0, \forall x \in [0, 1]$.

Clearly, 0 is a unique fixed point of T^2 and also of T.

It is easy to verify that the inequality of Theorem-2 is satisfied by T^2 for m=2 but not by T.

Following is the more general theorem :

THEOREM-3 : If T_1 and T_2 be two mappings on a compact metric space (X, d) into itself such that for some positive integers r, s, T_1^r and T_2^s satisfy the inequality

$$d (T_1^r x, T_2^S y) < q_1 d (x, y) + q_2 d (x, T_1^r x) + q_3 d (y, T_2^s y) +$$

$$q_4 d(x, T_2^S y) + q_5 d (y, T_1^r x)$$

for all x, y $\in$ X with $x \neq y$, where q_i (i = 1, 2, . . . 5) $\geqslant 0$, $\sum_{i=1}^{5} q_i = 1$,

then T_1 and T_2 have a unique common fixed point in X.

PROOF : Let $x \in X$ be an arbitrary element. We now define a sequence $\{ x_n \} \subseteq X$ as follows :

$$x_1 = T_1^r x, \; x_2 = T_2^S x_1, \; x_3 = T_1^r x_2, \ldots\ldots\ldots, x_{2n} = T_2^S x_{2n-1},$$

$$x_{2n+1} = T_1^r x_{2n}, \ldots\ldots\ldots$$

Since, X is compact and $x_n \in X$, $\forall n \geqslant 1$, we can always select a subsequence $\{ x_{n_i} \}$ from the sequence $\{ x_n \}$ such that $\lim_{i \to \infty} x_{n_i} = u$ for some $u \in X$.

If possible, suppose that $T_1^r u \neq u$.

Then $d (u, T_1^r u) \leqslant d (u, x_{2n_i}) + d (x_{2n_i}, T_1^r u)$

$$= d (u, x_{2n_i}) + d (T_1^r u, T_2^S x_{2n_i - 1})$$

$$< d (u, x_{2n_i}) + q_1 d (u, x_{2n_i - 1}) + q_2 d (u, T_1^r u)$$

$$+ q_3 d (x_{2n_i - 1}, T_2^S x_{2n_i - 1}) + q_4 d (u, T_2^S x_{2n_i - 1}) + q_5 d (x_{2n_i - 1}, T_1^r u)$$

$= d(u, x_{2n_i}) + q_1 d(u, x_{2n_i - 1}) + q_2 d(u, T_1^r u) + q_3 d(x_{2n_i - 1}, x_{2n_i}) +$

$q_4 d(u, x_{2n_i}) + q_5 d(x_{2n_i - 1}, T_1^r u).$

Now proceeding to the limit as $i \to \infty$, we have

$(1 - q_2 - q_5)\, d(u, T_1^r u) < 0$ —— which is a contradiction.

Hence, we have $T_1^r u = u$.

Likewise, we can show that $T_2^s u = u$.

We now show the unicity of the fixed point u.

Suppose, now, that $p\,(\neq u) \in X$ is another common fixed point of T_1^r and T_2^s both.

Then

$$d(p, u) = d(T_1^r p, T_2^s u) < q_1 d(p, u) + q_2 d(p, T_1^r p) + q_3 d(p, T_2^s p)$$
$$+ q_4 d(u, T_2^s p) + q_5 d(p, T_1^r u)$$
$$= (q_1 + q_4 + q_5)\, d(p, u) = (1 - q_2 - q_3)\, d(p, u)$$

$\Rightarrow (q_2 + q_3)\, d(p, u) < 0$ —— which is a contradiction, and thereby yielding $p = u$.

Consequently, u is a common fixed point of T_1^r and T_2^s in X.

Now, since, $T_1^r u = u$, we have $T_1 u = T_1(T_1^r u) = T_1^r (T_1 u)$ –

this implies that $T_1 u$ is another fixed point of T_1^r in X, but T_1^r has only one fixed point u in X.

Hence, this enables to conclude that $T_1 u = u$.

Similarly, we can show that $T_2 u = u$.

If z is another common fixed point of T_1 and T_2 in X, then clearly it is a common fixed point of T_1^r and T_2^s in X also. This yields $u = z$.

Hence, u is the unique common fixed point of T_1 and T_2 in X.
This completes the proof of the theorem.

§ 3 :- Let (X,d) be a metric space. Let $T : X \to X$ be a mapping which is said to possess the property (A) if it satisfies

$d(Tx,Ty) \leqslant ad(x,Tx) + bd(y,Ty) + cd(x,y)$

for all $x,y \in X$, $a,b,c \geqslant 0$, $a+b+c \leqslant 1$.

According to Kannan (1972) [5], an operator $T : X \to X$ is said to have the property (B) on $E \subset X$ if for every closed subset F of E containing more than one element and is self mapped by T, there exists an $x \in F$ such that

$$d(x, Tx) < \sup_{y \in F} d(y, Ty).$$

Kannan (1972) [5] studied this property (B) of an operator T in connection with the existence of fixed point. With the above notions, M. K. Chakraborty [2] proved the following

THEOREM-C :-

Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping having properties (A) and (B) over X. Let for any non-empty subset E of X which is mapped into itself by T, $p_n \to p$ imply $Tp_n \to p$ for any sequence $\{p_n\} \subset E$. Then T has a fixed point in X provided that $b \neq 1$. The fixed point is unique if $c \neq 1$.

Let (X, d) be a metric space. An operator $T : X \to X$ is said to satisfy Hardy and Regers [4] type of contraction if $d(Tx, Ty) \leqslant q_1 d(x, y) + q_2 d(x, Tx) + q_3 d(y, Ty) + q_4 d(x, Ty) + q_5 d(y, Tx)$ holds for all $x, y \in X$ where q_i ($i = 1, 2, 3, \ldots, 5$) $\geqslant 0$, $\sum_{i=1}^{5} q_i < 1$. ---(i)

We shall prove the following theorem in fixed point in compact metric space by weakening the condition (i) above, by allowing $\sum_{i=1}^{5} q_i = 1$ -- (ii) which extends the theorem-C.

We shall call (i) and (ii) together the property ($*$).

THEOREM :-1 :Let (X, d) be compact metric space and let T be a self mapping defined on X satisfying the properties ($*$) and (B) over X. Then if T be such that for any non-empty subset E of X, mapped into itself by T, $p_n \to p$ imply $Tp_n \to p$ where $\{ p_n \} \subset E$, then T has a fixed point in X. This fixed point will be unique provided that $q_2 > o$ or $q_3 > o$.

PROOF : We consider the space X(K) of all non-empty closed subsets K of X for which $T : K \to K$.

We now define a partial ordering in X(K) by the following rule :

$$K_{\alpha_1} < K_{\alpha_2}, \; K_{\alpha_1} \underset{\neq}{\supset} K_{\alpha_2}.$$

With this definition, the space X(K) will be a partially ordered set and hence by Kuratowski-Zorn lemma, there must exists a minimal element in X(K) which will be non-empty closed and invariant under T. Let it be denoted by K.

We now propose to show that K contains only one element.

If possible, suppose, K consists of more than one element. Then by the property (B), there exists an $x_o \in K$ such that $d (x_o, Tx_o) = r < \sup_{y \in K} d (y, Ty)$ .... (1)

Now consider the set $K_1 = \{ x \in K : d (x, Tx) \leqslant r \}$. Then by (1), K_1 is a non-empty proper subset of K. We shall first show that $T : K_1 \to K_1$. Take any $x \in K_1$.

Now, $d (Tx, T^2 x) = d (Tx, TTx) \leqslant q_1 d (x, Tx) + q_2 d (x, Tx) + q_3 d (Tx, TTx) + q_4 d (x, TTx) + q_5 d (Tx, Tx)$

$\Rightarrow (1-q_3-q_4)\, d(Tx, T^2x) \leqslant (1-q_3-q_5)\, d(x, Tx).$ ....(2)

By symmetry, we can write

$(1-q_2-q_5)\, d(Tx, T^2x) \leqslant (1-q_2-q_4)\, d(x, Tx).$ ....(3)

Adding (2) and (3) and cancelling the non-negative constant terms form both sides, we get

$d(Tx, T^2x) \leqslant d(x, Tx) \leqslant r \Rightarrow Tx \in K_1.$

This shows that K_1 is mapped into itself by T.

We shall now show that K_1 is closed.

Choose any sequence $\{s_n\} \subset K_1$ such that $s_n \to s$ as $n \to \infty$ where $s \in K$.

Hence by assumsion, $Ts_n \to s$.

Now, $d(s, Ts) \leqslant d(s, Ts_n) + d(Ts_n, Ts)$

$$\leqslant d(s, Ts_n) + q_1 d(s_n, s)\, q_2 + d(s_n, Ts_n) + q_3 d(s, Ts) + q_4 d(s_n, Ts) + q_5 d(s, Ts_n)$$

$$\leqslant (1 + q_5)\, d(s, Ts_n) + q_1 d(s_n, s) + q_2 r + q_3 d(s, Ts) + q_4 d(s_n, Ts .$$

Now passing to the limit as $n \to \infty$ we have after rearrangement

$(1-q_3-q_4)\, d(s, Ts) \leqslant q_2 r,$ ....(4)

In a similar way we can write by interchanging the roles of s_n and s,

$(1-q_2-q_5)\, d(s,Ts) \leqslant q_3 r.$ ....(5)

Adding these two we get

$\{2-(q_2+q_3+q_4+q_5)\}\, d(s,Ts) \leqslant (q_2+q_3)\, r$

$$\Rightarrow d(s,Ts) \leqslant \frac{(q_2+q_3)}{\{2-(q_2+q_3+q_4+q_5)\}}\, r \leqslant r, \Rightarrow s \in K_1.$$

Hence K_1 is closed.

Thus K_1 is a non-empty closed proper subset of K with $T : K_1 \to K_1$ and so $K_1 \in X(K)$. This contradicts the minimality of K in X (K). Hence K contains only one point, which is a fixed point of T in X.

Now suppose that $u \in K$ is this fixed point of T.

Let $v\ (\neq u) \in K$ be another fixed point of T. Then

$$d(u,v) = d(Tu,Tv) \leqslant q_1 d(u,v) + q_2 d(u,Tu) + q_3 d(v,Tv) + q_4 d(u,Tv) + q_5 d(v,Tu)$$

$$\leqslant (1 - q_2 - q_3)\, d(u,v),$$

$$\Rightarrow (q_2 + q_3)\, d(u,v) \leqslant 0 \Rightarrow d(u,v) = 0 \text{ as } q_2 > 0 \text{ or } q_3 > 0,$$

$$\Rightarrow u = v.$$

Consequently, T has a unique fixed point in X.

This completes the proof of the theorem.

NOTE : It is to be noted here that if we put $q_4 = 0, q_5 = 0, q_2 = a, q_3 = b$ and $q_3 = c$ then the theorem-C follows at once.

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