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ON SOME CLASSES OF STARLIKE AND CONVEX FUNCTIONS

S. K. Chatterjea and Abir Kumar Adhikary

1. **Introduction** : Let S denote the class of functions of the form

(1.1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are anatytic and univalent for $|z| < 1$. Then a necessary and sufficient condition for the function $w = f(z)$ to map $|z| < 1$ onto a star-shaped domain (i. e. f is starlike) is that

(1.2) $$\text{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant 0, \quad |z| < 1,$$

which implies that $|a_n| \leqslant n$; $n = 2, 3, \ldots$ Since the function $z/(1-z)^2 = z + 2z^2 + 3z^3 + \ldots + nz^n + \ldots$ maps $|z| < 1$ onto a star-shaped domain, the inequalities $|a_n| \leqslant n$ are sharp. We denote by S* the class of analytic, univalent and starlike functions in the unit disk.

On the other hand, a necessary and sufficient condition for the function $w = f(z)$ to map $|z| < 1$ onto a convex domain is that

(1.3) $$\text{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} \geqslant 0, \quad |z| < 1,$$

which implies that $w = f(z)$ maps $|z| < 1$ onto a convex domain if and only if $z f'(z)$ maps $|z| < 1$ onto a star-shaped domain and furthermore $|a_n| \leqslant 1$, $n = 2, 3, \ldots$. Since the function $z/(1-z) = z + z^2 + z^3 + \ldots$ maps $|z| < 1$ onto the half-plane $\text{Re}\{w\} > -\frac{1}{2}$, the inequalities $|a_n| \leqslant 1$ are sharp.

In the case of a convex function, the $\frac{1}{4}$ – theorem of Bieberbach for analytic and univalent function $w = f(z)$ can be improved as follows :

If the function $w = f(z)$ maps $|z| < 1$ onto a convex domain D, then D contains the disk $|w| < \frac{1}{2}$.

We denote by K the class of analytic, univalent and convex functions in the unit disk.

If we now consider a number α $(0 \leqslant \alpha < 1)$ and denote by S^*_α the class of those functions $f(z) \in S^*$ such that

$$(1.4)\quad \operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha, \quad |z| < 1,$$

then $f(z) \in S^*_\alpha$ is said to be starlike of order α.

A subclass $S^*_{[\alpha]}$ of S^*_α consisting of those $f(z)$ for which

$$(1.5)\quad \left| z\frac{f'(z)}{f(z)} - 1 \right| < 1-\alpha, \quad |z| < 1,$$

was considered by C. P. Mc Carty [1].

We can similarly define the class K_α of analytic, univalent and convex functions of order α.

Let us now denote by S (n) the class of functions.

$$(1.6)\quad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \geqslant 1),$$

that are analytic and univalent in the unit disk $|z| < 1$. Then f is said to be starlike of order α $(0 \leqslant \alpha < 1)$, denoted by $f \in S^*_\alpha(n)$, if $\operatorname{Re}\left\{z\frac{'(z)}{f(z)}\right\} > \alpha$, $(|z| < 1)$.

A Subclass $S^*_{[\alpha]}(n)$ of $S^*_\alpha(n)$ consists of those functions f (z) for which

$$(1.7)\quad \left| z\frac{f'(z)}{f(z)} - 1 \right| < 1-\alpha, \quad |z| < 1.$$

Similarly f, defined by (1.6), is said to be convex of order α $(0 \leqslant \alpha < 1)$, denoted by $f \in K_\alpha(n)$ if

$$(1.8)\quad \operatorname{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \alpha, \quad |z| < 1.$$

One can similarly define $K_{[\alpha]}(n)$.

Let T be the subclass of functions, defined by (1.6), consisting of functions expressible of the form

(1.9) $$f(z) = z - \sum_{k=n+1}^{\infty} |a_k| z^k, \quad n \geq 1,$$

then $T^*_\alpha(n)$ and $C_\alpha(n)$ are defined respectively as the subclasses of T that are starlike of order α and convex of order α. One can similarly define $T^*_{[\alpha]}(n)$ and $C_{[\alpha]}(n)$. It may be noted that class $S^*_\alpha(1)$ is same as S^*_α. Recently M. S. Kasy [2] has considered the class $S^*_\alpha(2)$ where $a_3 = c$ is fixed and $|c| < 1$. Also H. Silverman [3] has considered the class $S^*_\alpha(1)$ and a subclass $T^*_\alpha(1)$ of $S^*_\alpha(1)$ where $f \in T^*_\alpha(1)$ is expressed in the form

(1.10) $$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

In a recent paper [4] the first author has considered some properties of the classes $S^*_\alpha(n)$, $S^*_{[\alpha]}(n)$ and $T^*_\alpha(n)$. Since the classes $S^*_\alpha(1)$ and $T^*_\alpha(1)$ are special cases of the classes $S^*_\alpha(n)$ and $T^*_\alpha(n)$ respectively, it is not appropriate [5] to derive any conclusion for $S^*_\alpha(n)$ or $T^*_\alpha(n)$ from that for $S^*_\alpha(1)$ or $T^*_\alpha(1)$ respectively by setting $a_2 = a_3 \cdots = a_n = o$, because n is not fixed, but on the other hand n is a positive integral variable in the definition of $S^*_\alpha(n)$ or $T^*_\alpha(n)$. Furthermore any property like representation formula, distortion theorem, etc. of $S^*_\alpha(1)$ or $T^*_\alpha(1)$ comes out as a special case from that of $S^*_\alpha(n)$ or $T^*_\alpha(n)$ respectively.

In the present paper we shall discuss some properties of the above classes of starlike and convex functions.

2. Some properties of the classes S^*_α and K_α

Let $f \in S^*_\alpha$. Then we have

$$\text{Re}\left\{\frac{zf'}{f}\right\} > \alpha. \quad |z| < 1$$

$$\text{or,}\ \text{Re}\left\{\frac{zf'}{f} - \alpha\right\} > 0.$$

Now by virtue of the relation (33) of [6, P. 169] we obtain

$$(2.1) \quad \frac{1-r}{1+r} \leqslant \left|\frac{zf'}{f} - \alpha\right| \leqslant \frac{1+r}{1-r}, \quad |z| = r < 1.$$

In particular, when $\alpha = 0$ we obtain from (2.1)

$$(2.2) \quad \frac{1-r}{r(1+r)} \leqslant \left|\frac{f'}{f}\right| \leqslant \frac{1+r}{r(1-r)}, \quad |z| = r < 1,$$

which is a well-known [7, P 5] distortion theorem for univalent function f.
Furthermore (2.1) implies that

$$(2.3) \quad \alpha - \frac{1+r}{1-r} \leqslant \text{Re}\left\{\frac{zf'}{f}\right\} \leqslant \alpha + \frac{1+r}{1-r}$$

and

$$(2.4) \quad \alpha + \frac{1-r}{1+r} \leqslant \text{Re}\left[\frac{zf'}{f}\right]; \quad \text{Re}\left[\frac{zf'}{f}\right] \leqslant \alpha - \frac{1-r}{1+r}$$

Since $\text{Re}\left\{\frac{zf'}{f}\right\} > \alpha$, we have from (2.3) and (2.4)

$$(2.5) \quad \alpha + \frac{1-r}{1+r} \leqslant \text{Re}\left[\frac{zf'}{f}\right] \leqslant \alpha + \frac{1+r}{1-r}.$$

In other words, we have

(2.6) $$\frac{1}{r}\left[\alpha+\frac{1-r}{1+r}\right]\leqslant\frac{\partial}{\partial r}\log|f(z)|\leqslant\frac{1}{r}\left[\alpha+\frac{1+r}{1-r}\right],$$

since $\operatorname{Re}\left(\frac{zf'}{f}\right)=r\frac{\partial}{\partial r}\log|f(z)|$.

Integrating (2.6) from r_1 to r_2, where $o<r_1<r_2<1$, we deduce

(2.7) $$\frac{r_2^{\alpha+1}}{(1+r_2)^2}\Big/\frac{r_1^{\alpha+1}}{(1+r_1)^2}\leqslant\left|\frac{f(r_2e^{i\theta})}{f(r_1e^{i\theta})}\right|\leqslant\frac{r_2^{\alpha+1}}{(1-r_2)^2}\Big/\frac{r_1^{\alpha+1}}{(1-r_1)^2}$$

Making $\alpha \doteq o$ and $r_1\to o$ in (2.7) we deduce a well-known [7] distortion theorem for univalent function f with $r=r_2$, viz.

(2.8) $$\frac{r}{(1+r)^2}\leqslant|f(z)|\leqslant\frac{r}{(1-r)^2}.$$

It therefore follows from (2.2) and (2.8) that

(2.9) $$\frac{1-r}{(1+r)^3}\leqslant|f'(z)|\leqslant\frac{1+r}{(1-r)^3},$$

which is also well-known [7] for univalent function f.

Next let $f\in K_\alpha$. Then we have

$$\operatorname{Re}\{1+\frac{zf''}{f'}\}>\alpha,\ |z|<1$$

$$\text{or } \operatorname{Re}\{\frac{zf''}{f'}+(1-\alpha)\}>o.$$

By virtue of the same relation (33) of [6, p. 169] we obtain

(2.10) $$\frac{1-r}{1+r}\leqslant\left|\frac{zf''}{f'}+(1-\alpha)\right|\leqslant\frac{1+r}{1-r},\ |z|=r<1.$$

Now (2.10) implies that

(2.11) $$(\alpha-1)-\frac{1+r}{1-r}\leqslant\operatorname{Re}\left\{\frac{zf''}{f'}\right\}\leqslant(\alpha-1)+\frac{1+r}{1-r}$$

and

(2.12) $(\alpha - 1 + \frac{1-r}{1+r} \leqslant \text{Re}\left\{\frac{z\ f''}{f'}\right\}$; $\text{Re}\left\{\frac{z\ f''}{f'}\right\} \leqslant (\alpha - 1) - \frac{1-r}{1+}$

Since $\text{Re}\left\{\frac{z f''}{f'} + 1\right\} > \alpha$, we have from (2 11) and (2.12)

(2.13) $$(\alpha - 1) + \frac{1-r}{1+r} \leqslant \text{Re}\left\{\frac{z\ f''}{f'}\right\} \leqslant (\alpha - 1) + \frac{1+r}{1-r}$$

In other words, we have

(2.14) $$\frac{1}{r}\left[(\alpha - 1) + \frac{1-r}{1+r}\right] \leqslant \frac{\partial}{\partial r} \log |f'(z)| \leqslant \frac{1}{r}\left[(\alpha - 1) + \frac{1+r}{1-r}\right],$$

since $\text{Re}\left\{\frac{z\ f''}{f}\right\} = r \frac{\partial}{\partial r} \log |f'(z)|$.

Integrating (2.14) from o to r, we obtain

(2.15) $$\frac{r^{\alpha}}{(1+r)^2} \leqslant |f'(z)| \leqslant \frac{r^{\alpha}}{(1-r)^2}$$

Now $|f(z)| = \left|\int_o^z f'(z)\, dz\right| \leqslant \int_o^r |f'(z)|\, dr \leqslant \int_o^{\alpha} \frac{r^{\alpha}}{(1-r)^2}$

To obtain a lower bound for $|f(re^{i\theta})|$, we assume without any loss of generality, that $f(re^{i\theta}) = Re^{i\phi}$, where $R < \frac{1}{2}$, since otherwise there is nothing to prove It then follows that the straight line segment λ from o to $Re^{i\phi}$ lies entirely in the image of $|z| < 1$ by f (z). Hence λ corresponds to a path L in $|z| < 1$, which joins $z = o$ to $re^{i\theta}$. Thus we have

$$R = \int_{\lambda} |dw| = \int_L \left|\frac{dw}{dz}\right| |dz| \geqslant \int_o^r \frac{r^{\alpha}}{(1+r)^2} dr.$$

Hence we have

(2.16) $$\int_o^r \frac{r^{\alpha}}{(1+r)^2} dr \leqslant |f(z)| < \int_o^r \frac{r^{\alpha}}{(i-r)^2} dr.$$

In particular, when $\alpha = 0$, we obtain from (2.15) and (2.16)

$$(2.17)\quad \frac{1}{(1+r)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1-r)^2}$$

and

$$(2.18)\quad \frac{r}{1+r} \leqslant |f(z)| \leqslant \frac{r}{1-r}$$

which are well-known [6, p. 225] more improved distortion theorems of convex univalent function f than the distortion theorems of univalent function f (or of starlike univalent function f by virtue of (2.8) and (2.9)).

Since we know that $K_0 \equiv S^*_{\frac{1}{2}}$, we have (2.17) and (2.18) for any $f \in S^*_{\frac{1}{2}}$, which are well-known results of Zhuo—ren Wu [8].

Again $f \in K_0 \Longleftrightarrow z f' \in S^*_0$, so that (2.17) follows at once from (2.8).

Similarly $f \in K_\alpha \Longleftrightarrow z f' \in S^*_\alpha$, so that (2.14) follows at once from (2.6) and consequently (2.15) and (2.16) can be rediscovcred.

3. Some properties of the classes $S^*_{[\alpha]}(n)$ and $K_{[\alpha]}(n)$

It may be noted that Ming—Po Chen [9] considered the class $\mathscr{A}$ of functions expressible of the form :

$$(3.1)\quad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,\ n \geqslant 1,$$

satisfying the condition

$$(3.2)\quad \left| \frac{z f'(z)}{f(z)} - 1 \right| < \alpha'$$

for a given α, $0 < \alpha \leqslant 1$, for $|z| \leqslant 1$. He remarked that every function of the above class $\mathscr{A}$ is a starlike function (of order zero), because in the introdnction he only introdnced starlike functions defined by the necessary and sufficient condition

$\mathrm{Re}\left\{\frac{zf'}{f}\right\} > 0$ for $|z| < 1$, but actually every function $\epsilon\ \mathcal{A}$ must be a starlike function of order $(1-\alpha)$. He derived the following properties for the class $\mathcal{A}$.

Property 1 (Representation theorem)

$f(z) \in \mathcal{A}$ if and only if

$$(3.3)\quad f(z) = z \exp\left(\int_0^z t^{n-1}\phi(t)\,dt\right),$$

where $\phi(z)$ is analytic and satisfies the condition

$$|\phi(z)| \leqslant \alpha,\ 0 < \alpha \leqslant 1, \text{ for } |z| < 1.$$

Property 2 (Distortion theorems)

For all $f(z) \in \mathcal{A}$ the following properties hold :

$$(3.4)\quad |z| \exp\{-\alpha(|z|^n)/_n\} \leqslant |f(z)| \leqslant |z| \exp\{\alpha(|z|^n)/_n\}$$

and

$$(3.5)\quad (1-\alpha|z|^n)\exp\{-\alpha(|z|^n)/_n\} \leqslant |f'(z)| \leqslant (1+\alpha|z|^n)\exp\{\alpha(|z|^n)/_n\}.$$

Now for the class $S^*_{[\alpha]}(n)$, we are to change α to $(1-\alpha)$ in the discussion of Ming-Po Chen, so that $\alpha \in [0, 1)$ and we obtain

Property 1′ $f(z) \in S^*_{[\alpha]}(n)$ if and only if

$$(3.6)\quad f(z) = z \exp\left[\int_0^z t^{n-1}\phi(t)\,dt\right],$$

where $\phi(z)$ is analytic and satisfies the condition

$$|\phi(z)| \leqslant 1-\alpha,\ \alpha \in [0, 1),\ |z| < 1.$$

Property 2′ If $f(z) \in S^*_{[\alpha]}(n)$, then

$$(3.7)\quad |z| \exp\{(\alpha-1)(|z|^n)/_n\} \leqslant |f(z)| \leqslant |z| \exp\{(1-\alpha)(|z|^n)/_n\}$$

and

$$(3.8)\quad (1+(\alpha-1)|z|^n)\exp\{(\alpha-1)(|z|^n)/_n\} \leqslant |f'(z)|$$
$$\leqslant (1+(1-\alpha)|z|^n)\exp\{(1-\alpha)(|z|^n)/_n\}.$$

The proof of Property 2′ can be shortened (unlike Ming-Po Chen) as follows ;

Since $\left| \int_o^z t^{n-1} \phi(t)\, dt \right|$

$$\leqslant \int_o^{|z|} |t|^{n-1} |\phi(t)| \; |dt|$$

$$\leqslant \int_o^{|z|} |t|^{n-1} (1-\alpha) |dt| = (1-\alpha)(|z|^n)/_n,$$

(3.7) follows by virtue of Property 1′.

Again (3.8) follows by applying the triangle inequality to $f'(z) = (1+z^n \phi(z)) f(z)/z$ and by (3. 7).

We now remark that our representation for $f \in S^*_{[\alpha]}(n)$ yields the following representation of Mc Carty for $S^*_{[\alpha]}(1)$ as a particular case

(3.9) $f \in S^*_{[\alpha]}(1) = z \exp \{ \int_o^z \phi(t)\, dt \},$

where $\phi(z)$ is analytic for $|z| < 1$ and $|\phi(z)| < 1-\alpha$ for $|z| < 1$ and $\alpha \in [o, 1)$,

which dearly shows that it is not possible to derive the representation of $S^*_{[\alpha]}(n)$ from that of $S^*_{[\alpha]}(1)$.

Again our distortion theorems for $f \in S^*_{[\alpha]}(n)$ yields the following distortion theorems of Mc Carty for $S^*_{[\alpha]}(1)$ as a particular case :

If $f(z) \in S^*_{[\alpha]}(1)$, then

(3.10) $|z|\, e^{(\alpha-1)|z|} \leqslant |f(z)| \leqslant |z|\, e^{(1-\alpha)|z|}$

and

(3.11) $(1 + (\alpha-1)|z|)\, e^{(\alpha-1)|z|} \leqslant |f'(z)|$

$$\leqslant (1 + (1-\alpha)|z|)\, e^{(1-\alpha)|z|},$$

which again reveals the fact that it is not possible to derive distortion theorems of $S^*_{[\alpha]}$ (n) from those of $S^*_{[\alpha]}$ (1).

Next we notice that $f \in K_{[\alpha]}(n) \Longleftrightarrow z f' \in S^*_{[\alpha]}(n)$,

so that we obtain from Property 1′ and Property 2′ the following properties :

Property 3 $f(z) \in K_{[\alpha]}(n)$ if and only if

$$(3.12) \quad f'(z) = \exp\left\{\int_0^z t^{n-1}\phi(t)\,dt\right\},$$

where $\phi(z)$ is analytic and satisfies the condition

$$|\phi(z)| \leqslant 1-\alpha, \quad \alpha \in [0, 1), \quad |z| < 1$$

Property 4 If $f(z) \in K_{[\alpha]}(n)$, then

$$(3.13) \quad \exp\{(\alpha-1)(|z|^n)/n\} \leqslant |f'(z)| \leqslant \exp\{(1-\alpha)(|z|^n)/n\}.$$

4. *Some properties of the classes $T^*_\alpha(n)$ and $C_\alpha(n)$*

Let $f \in T^*_\alpha(n)$. Then from the result of [5, p 117) we have the following distortion.

$$(4.1) \quad |z| - \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^{n+1} \leqslant |f(z)| \leqslant |z| + \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^{n+1}$$

Now we notice that $f \in C_\alpha(n) \Longleftrightarrow z f' \in T^*_\alpha(n)$, so that we at once obtain from (4.1)

$$(4.2) \quad 1 - \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^n \leqslant |f'(z)| \leqslant 1 + \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^n,$$

which is (2.12) in the work [5, p. 119].

The above method of deriving (4.2) indicates that (2.12) of [5, p 119] need not be derived independently and moreover Lemma 2 of the first author of the present paper, which is utilized in deriving (2.12) of [5, p.119], follows immediately from Lemma 1 of the first author of the present paper by using the fact $f \in C_\alpha(n) \Longleftrightarrow z f' \in T^*_\alpha(n)$, which was already pointed out by the first author of the present paper in [4].

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Dept. of Pure Mathematics
Calcutta University
Calcutta-700 019, India