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ON SOME CLASSES OF UNIVALENT FUNCTIONS

ABIR KUMAR ADHIKARY & S. K. CHATTERJEA

1. G. S. Salagean [4] defined the classes $S_n(\alpha)$ of univalent and normalized functions f in the unit disc $U=\{z\in C\ ;\ |z|<1\}$ by

$$S_n(\alpha)=\left\{f : \operatorname{Re}\frac{D^{n+1}f(z)}{D^nf(z)}>\alpha,\ z\in U\right\},$$

where $n\in N_o=N\cup\{o\}$ and $\alpha\in[o, 1)$,

by means of the following differential operators :
$D^of(z)=f(z),\ D^1f(z)\equiv Df(z)=zf'(z),\ D^nf(z)=D(D^{n-1}f(z))$.

It may be of interest to observe that his $S_o(\alpha)$ is the well known class S^*_α of starlike functions of order α and his $S_1(\alpha)$ is the well known class $K\alpha$ of convex functions of order α. Furthermore $S_1(\alpha)\subset S_0(\alpha)$ by virtue of his result $S_{n+1}(\alpha)\subset S_n(\alpha)$. Now for Salagean classes $S_n(\alpha)$ one can deduce that (cf. [1])

$$(1.1)\quad \alpha+\frac{1-r}{1+r}\leqslant \operatorname{Re}\frac{D^{n+1}f(z)}{D^nf(z)}\leqslant \alpha+\frac{1+r}{1-r},\quad |z|=r<1.$$

since
$$\frac{D^{n+1}f(z)}{D^nf(z)}=\frac{D(D^nf(z)}{D^nf(z)}=\frac{z\,(D^nf(z))'}{D^nf(z)},$$

$$\operatorname{Re}\left\{\frac{z\,(D^nf(z))'}{D^nf(z)}\right\}=r\frac{\partial}{\partial r}\log|D^nf(z)|.$$

It therefore follows from (1.1) that

(1.2] $$\frac{1}{r}\left(\alpha+\frac{1-r}{1+r}\right)\leqslant\frac{\partial}{\partial r}\log|D^n f(z)|\leqslant\frac{1}{r}\left(\alpha+\frac{1+r}{1-r}\right).$$

In particular, when r=o, one obtains from (1.2)

(1,3) $$\frac{1}{r}\left(\alpha+\frac{1-r}{1+r}\right)\leqslant\frac{\partial}{\partial r}\log|f(z)|\leqslant\frac{1}{r}\left(\alpha+\frac{1+r}{1-r}\right),$$

which was obtained by us in a previous work [1].

Again, when n=1, one obtains from (1.2)

(1.4) $$\frac{1}{r}\left\{(\alpha-1)+\frac{1-r}{1+r}\right\}\leqslant\frac{\partial}{\partial r}\log|f'(z)|\leqslant\frac{1}{r}\left\{(\alpha-1)+\frac{1+r}{1-r}\right\}$$

Integrating (1.4) from o to r, one obtains

(1.5) $$\frac{r^\alpha}{(1+r)^2}\leqslant|f'(z)|\leqslant\frac{r^\alpha}{(1+r)^2},$$

from which one can derive

(1.6) $$\int_o^r\frac{r^\alpha}{(1+r)^2}\,dr\leqslant|f(z)|\leqslant\int_o^r\frac{r^\alpha}{(1-r)^2}\,dr,$$

which was also obtained by us in the previous work [1].

Since Salagean remarked that all functions in $S_n(\alpha)$, $n\in N_o$, $\alpha\in[o,1)$ are starlike and all functions in $S_n(\alpha)$, $n\in N$, $\alpha\in[o,1)$ are convex, we can remark that the same distortion theorem (1.2) is true for starlike functions and convex functions according as $n\in N_o$ and $n\in N$. It is evident that $f\in S_n\,\alpha)$, $n\in N\Rightarrow z f'\in S_n(\alpha)$, $n\in N_o$.

Next we shall show that the co-efficient result in connection with $S_n(\alpha)$ in theorem 4 of Salagean follows at once from that in connection with S^*_α due to H. Silverman and E. M. Silvia [4]. In fact, we notice that

$$S_n(\alpha)=\left\{f:\operatorname{Re}\frac{z\,\phi'(z)}{\phi(z)}>\alpha,\ z\in U\right\},$$

where $\phi(z)\equiv D^n f(z)$.

Now $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and therefore

$$\phi(z) \equiv D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \in S^*_{\alpha}, \; n \in N_0.$$

Let us put $\Phi = z + \sum_{j=2}^{\infty} b_j z^j$, where $b_j = j^n a_j$ and $b_2 = 2^n a_2$ ($|a_2| = a$). Then from the coefficient result of Silverman and Silvia we get

$$|b_j| \leqslant \frac{1 + |b_2|}{3 - 2\alpha} \frac{\prod_{k=2}^{j} (k - 2\alpha)}{(j-1)!}, \; j = 3, 4, \ldots$$

$$\text{i.e. } |a_j| \leqslant \frac{1 + 2^n a}{3 - 2\alpha} \frac{\prod_{k=2}^{j} (k - 2\alpha)}{(j-1)! \, j^n}$$

which is the coefficient result derived by Salagean who offered a proof in about two pages. In fact, putting $n = o$ and $n = 1$ successively in the above coefficient result of Silverman and Silvia for $S^*(\alpha)$ and K_{α} respectively.

2. S Ruscheweyh [3] defined another classes K_n of univalent and normalized function f in the unit disc $U = \{z \in C ; |z| < 1\}$ by

$$K_n = \left\{ f ; \operatorname{Re} \frac{\mathfrak{D}^{n+1} f(z)}{\mathfrak{D}^n f(z)} > \tfrac{1}{2}, \; z \in U \right\},$$

where $n \in N_0 = N \cup \{o\}$ and $\mathfrak{D}^n f = z\,(z^{n-1} f)^{(n)} / n!$.
It may be of interest to observe that his K_0 is the well known class $S^*_{\frac{1}{2}}$ of starlike functions of order $\frac{1}{2}$ and his K_1 is the well known class K of convex functions of order zero.
Thus $K_1 \equiv K \subset K_0$ is a special case of the following result of Ruscheweyh

$$K_{n+1} \subset K_n, \; n \in N_0$$

Now for Ruscheweyh classes K_n one can deduce that (cf. [1])

$$(2.1)\qquad \tfrac{1}{2}+\frac{1-r}{1+r}\leqslant \operatorname{Re}\frac{\mathfrak{D}^{n+1}f}{\mathfrak{D}^{n}f}\leqslant -\frac{1}{2}+\frac{1+r}{1-r},\quad |z|=r<1$$

Since we know that [2]

$$\frac{\mathfrak{D}^{n+1}f}{\mathfrak{D}^{n}f}=\frac{1}{n+1}\left(z\frac{(\mathfrak{D}^{n}f)'}{\mathfrak{D}^{n}f}+n\right),$$

we have

$$\operatorname{Re}\frac{\mathfrak{D}^{n+1}f}{\mathfrak{D}^{n}f}=\frac{1}{n+1}\left(n+r\frac{\partial}{\partial r}\log|\mathfrak{D}^{n}f|\right).$$

It therefore follows from (2.1) that

$$(2.2)\qquad \tfrac{1}{2}+\frac{1-r}{1+r}\leqslant\frac{1}{n+1}\left(n+r\frac{\partial}{\partial r}\log|\mathfrak{D}^{n}f|\right)\leqslant\tfrac{1}{2}+\frac{1+r}{1-r}.$$

In particular, when n=o, obtains from (2.2)

$$(2.3)\qquad \frac{1}{r}\left(\tfrac{1}{2}+\frac{1-r}{1+r}\right)\leqslant\frac{\partial}{\partial r}\log|f(z)|\leqslant\frac{1}{r}\left(\tfrac{1}{2}+\frac{1+r}{1-r}\right),$$

which is a particular case of our previous result [1].

Again, when n=1, one obtains from (2.2)

$$(2.4)\qquad \frac{1}{r}\left(-\tfrac{1}{2}+\frac{1-r}{1+r}\right)\leqslant\frac{\partial}{\partial r}\log|f'(z)|\leqslant\frac{1}{r}\left(-\tfrac{1}{2}+\frac{1+r}{1-r}\right).$$

Integrating (2 4) from o to r one obrains

$$(2.5)\qquad \frac{\sqrt{r}}{(1+r)^2}\leqslant|f'(z)|\leqslant\frac{\sqrt{r}}{(1-r)^2},$$

from which one can derive

$$(2.6)\qquad \int_0^r\frac{\sqrt{r}}{(1+r)^2}\,dr\leqslant|f(z)|\leqslant\int_0^r\frac{\sqrt{r}}{(1-r)^2}\,dr$$

i. e. $\tan^{-1}\sqrt{r}-\dfrac{\sqrt{r}}{1+r}\leqslant|f(z)|\leqslant\dfrac{\sqrt{r}}{1-r}-\tfrac{1}{2}\log\dfrac{1+\sqrt{r}}{1-\sqrt{r}}.$

It may be noted that (2.5) and (2.6) are well compared with the following known results for convex functions of order zero

$$(2.7)\quad \frac{1}{(1+r)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1-r)^2}$$

$$(2.8)\quad \frac{r}{1+r} \leqslant |f(z)| \leqslant \frac{r}{1-r}$$

Like §1 we can remark that all functions in K_n, $n \in N_o$, are starlike of order $\frac{1}{2}$ and all functions in K_n, $n \in N$, are convex of order zero and therefore the same distortion theorem (2.2) is true for starlike functions of order $\frac{1}{2}$ and convex functions of order zero according as $n \in N_o$ and $n \in N$.

Finally we shall point out the geometrical structure of the elements in K_n. To this end, we notice that

$$K_n = \{ f : \mathrm{Re} \left[\frac{1}{n+1} (n + \frac{z(\mathfrak{D}^n f)'}{\mathfrak{D}^n f}) \right] > \tfrac{1}{2},\ z \in U \}$$

$$\text{i. e. } K_n = \{ f : \mathrm{Re} \frac{z\Phi'(z)}{\Phi(z)} > \frac{1-n}{2},\ z \in U \},$$

where $\Phi = \mathfrak{D}^n f$.

Now $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and therefore

$$\Phi(z) \equiv \mathfrak{D}^n f(z) = [\, z(1-z)^{-n-1}] * f(z)$$

$$= z + \sum_{j=2}^{\infty} \frac{(n+1)_{j-1}}{(j-1)!} a_j z^j \in S^*_{(1-n)/2}, \quad n \in N_o,$$

although $S^*_{(1-n)/2}$ is not known in the literature except for $n = o$ and $n = 1$. Indeed, when $n = o$. $\Phi \equiv f \in S^*_{\frac{1}{2}}$ and when $n = 1$, $\Phi \in S^*_0$ or in other words $f \in K$. Yet Ruscheweyh defined the class K_{-1} as the set of functions f with $\mathrm{Re}\ f(z)/z > \frac{1}{2}$, $z \in U$. Actually when $n = -1$, $\Phi \in S_1{}^*$, which is also not known in the literature. So the notation $S^*_{(n-1)/2}$ is not unnatural in the geometric function theory — which furnishes an open problem in this paper. When n is replaced by an arbitrary real number $\alpha > -1$, it may be of interest to observe that $\Phi(z) \equiv \mathfrak{D}^{\alpha} f(z) \in S^*_{(1-\alpha)/2}$.

REFERENCES

[1] Chatterjea, S. K. and Adhikary Abir Kumar — On some classes of starlike and convex functions, this Journal vol 6 (1987-88), 63.

[2] Fukui, S. and Sakaguchi, K. — An extension of a theorem of S. Ruscheweyh, Bull. Fac. Edu. Wakayama Univ. Nat. Sci. 29 (1980), 1—

[3] Ruscheweyh, Stephen — New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975); 109—115

[4] Salagean, Grigore Stefan—Subclasses of univalent functions, Lecture Notes in Math. 1013 (Springer—Verlag (1983)), 362--372.

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Dept. of Pure Mathematics
Calcutta University
Calcutta-700 019, India