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QUASI INNER PRODUCT FOR SOME BILINEAR GENERATING FUNCTIONS OF LEGENDRE POLYNOMIALS

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We recall the method of quasi inner product, introduced by L.R. Bragg [1], which is very convenient for developing results for some bilinear generating functions of Legendre polynomials.

First we observe that

$$(1) \quad \sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2tx+t^2)^{-\frac{1}{2}} \equiv f(x,t)$$

and

$$(2) \quad \sum_{n=0}^{\infty} P_n(y)\, t^n = (1-2ty+t^2)^{-\frac{1}{2}} \equiv f(y,t),$$

where $|t| < 1$.

Then by Bragg's method we have

$$f(x,t) \circ f(y,t) = \sum_{n=0}^{\infty} P_n(x)\, P_n(y)\, t^{2n}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (1-2xte^{i\theta}+t^2e^{2i\theta})^{-\frac{1}{2}} (1-2yte^{-i\theta}+t^2e^{-2i\theta})^{-\frac{1}{2}}\, d\theta$$

$$(3) \quad = \frac{1}{2\pi} \int_0^{2\pi} [\,1+t^4+4t^2xy+2t^2\cos 2\theta - 2t(1+t^2)(x+y)\cos\theta - 2it(1-t^2)(x-y)\sin\theta\,]^{-\frac{1}{2}}\, d\theta,$$

which may be compared with the results of Legendre [2], Watson [3], Maximon [4] and Carlitz [5] for the generating series

$$\sum_{n=0}^{\infty} P_n(x) P_n(y) t^n.$$

In particular, we have from (3) on using $y=x$

$$\sum_{n=0}^{\infty} [P_n(x)]^2 t^{2n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} [1 + t^4 + 4t^2x^2 + 2t^2\cos 2\theta - 4xt(1+t^2)\cos\theta]^{-\frac{1}{2}} d\theta,$$

which is a special case of Bragg's result [1, formula (4.7)].

Again using $y=1$ in (3) and comparing it with (1) we get the following integral evaluation

$$(5) \quad \frac{1}{2\pi}\int_0^{2\pi} (1 - te^{-i\theta})^{-1} (1 - 2xte^{i\theta} + t^2e^{2i\theta})^{-\frac{1}{2}} d\theta$$

$$= (1 - 2t^2x + t^4)^{-\frac{1}{2}},$$

which includes the result (when $x=1$) :

$$(6) \quad \int_0^{2\pi} \frac{d\theta}{1 - 2t\cos\theta + t^2} = \frac{2\pi}{1-t^2},$$

which also follows from (4) on using $x=1$.

Again if we observe that

$$(7) \quad \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) P_n(x) t^n = \tfrac{1}{2}(1-t^2)(1-2tx+t^2)^{-\frac{3}{2}}$$

and

$$(8) \quad \sum_{n=0}^{\infty} P_n(y) t^n = (1 - 2ty + t^2)^{-\frac{1}{2}},$$

where $| t < 1$,

then by Bragg's method we have

$$(9) \quad \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) P_n(x) P_n(y) t^{2n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-t^2e^{2i\theta})(1-2tye^{-i\theta}+t^2e^{-2i\theta})}{2[(1-2txe^{i\theta}+t^2e^{2i\theta})(1-2tye^{-i\theta}+t^2e^{-2i\theta})]^{\frac{3}{2}}} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-t^4-2it^2\sin 2\theta+2t^3y(\cos\theta+\sin\theta)-2ty(\cos\theta-i\sin\theta)\,d\theta}{2[1-2t(xe^{i\theta}+ye^{-i\theta})+2t^2\cos 2\theta-2t^3(xe^{-i\theta}+ye^{i\theta})+4t^2xy+t^4]^{\frac{3}{2}}},$$

which may be compared with the result of Watson [3].

In particular, we have from (9) on using $y=x$

$$(10)\quad \sum_{n=0}^{\infty}(n+\tfrac{1}{2})P^{2}{}_{n}(x)\,t^{2n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-t^2)(1+t^2-2tx\cos\theta)\,d\theta}{2[(1+t^2-2tx\cos\theta)^2-4t^2(1-x^2)\sin^2\theta]^{\frac{3}{2}}}$$

Again using $y=1$ in (9) and comparing it with (7) we obtain

$$(11)\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-t^2e^{2i\theta})\,d\theta}{(1-te^{-i\theta})(1-2txe^{i\theta}+t^2e^{2i\theta})}$$

$$=(1-t^4)(1-2t^2x+t^4)^{-\frac{3}{2}},$$

which includes the result (when $x=1$)

$$(1.2)\quad \int_0^{2\pi}\frac{d\theta}{(1-2t\cos\theta+t^2)^2} = \frac{2\pi(1+t^2)}{(1-t^2)^3}$$

which also follows from (10) on using $x=1$, by virtue of the fact

$$\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\,t^{2n} = \frac{1}{2}\,\frac{(1+t^2)}{(1-t^2)^2}.$$

REFERENCES

[1] L. R. Bragg—SIAM J. MATH. ANAL. 17 (1986), 220-230

[2] E. T. Whittaker & G.N. Waston—A course of modern analysis, Combridge Univ. Press (1952), p. 335

[3] G. N. Watson—J. Lond. Math. Soc. 8 (1933), 289-292

[4] G. Maximon—Norske Videnskabers Selskab Forkandlinger 20 (1957), 82-86

(5) L. Carlitz—Boll. Un. Mat. Ital (3), XIV (1959), 6-9

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