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GENERAL FIXED POINT PRINCIPLE

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The general fixed point principle is based on the following two simple conditions :

(H1) : The operator $T : \bar{G} \to X$ is compact on the closure of G, wherc G is a non-empty open bounded subset of the real Banach space X

and

(H2) : The image set $T(\partial G)$ contains no points of exterior rays i.e., there is an elmeent $x_o \in G$ such that

$T(x) \neq x_o + \beta(x - x_o)$ for all $x \in \partial G$ and $\beta > 1$.

In 1986, E. Zeidler [1] has proved the following theorem by the application of fixed point index :

THEO EM : 1 : (Zeidler [1]) : If (H1) and (H2) be satisfied, then T has a fixed point in G.

The importance of the condition (H2) lies in the fact that either of the following well-known conditions implies (H2) :

(a) *Rothe condition :* The set G is convex and $T(\partial G) \subseteq \bar{G}$.

(b) *Leray-Schauder condition :* $0 \in G$ and $tTx \neq x$ for all $(x, t) \in \partial G \times (0,1)$.

(c) *Altman condition :* There is an $x_o \in G$ such that

$$\| Tx - x \|^2 \geqslant \| Tx - x_o \|^2 - \| x - x_o \|^2 \text{ for all } x \in \partial G.$$

REMARK : Let (H1) be given The condition (H2) will be satisfied and thus T will have a fixed point in G if the followiag condition is satisfied :

(H3) Let $0 \in G$ and $\| x - T(x) \| \geqslant \| T(x) \|$ for all $x \in \partial G$.

The object of this paper is to prove the following fixed point theorem by using the conditions (H1) and (H3) :

THEOREM : 2 If (H1) and (H3) be satisfied, then T will have a fixed point in G.

Proof of Theorem : 2 We shall merely show that the condition (H2) will be satisfied if (H3) holds.

Indeed, we take $x_0 = 0 \in G$, then (H2) becomes $T(x) \neq \beta x$ for all $x \in \partial G$ and $\beta > 1$.

If possible let, $T(x) = \beta x$ with $x \in \partial G$ and β is a real number.

Then from (H3) we get

$$\| x - T(x) \| \geqslant \| T(x) \| \text{ for all } x \in \partial G$$

$$\Rightarrow \quad \| x - \beta x \| \geqslant \| \beta x \|$$

$$\Rightarrow \quad \| x - \beta x \|^2 \geqslant \| \beta x \|^2$$

$$\Rightarrow \quad (1 - \beta)^2 \| x \|^2 \geqslant \beta^2 \| x \|^2$$

$$\Rightarrow \quad (1 - \beta)^2 \geqslant \beta^2$$

$$\Rightarrow \quad \beta \leqq \tfrac{1}{2}, \text{ which contradicts (H2).}$$

Hence $T(x) \neq \beta x$ for all $x \in \partial G$ and $\beta > 1$ i.e., we can say that (H2) will be satisfied if (H3) holds and consequently T will have a fixed point in G.

REFERENCE

[1] *E. Zeidler :* Non-linear Functional Analysis & Applications : Fixed Point Theorems I; (1986) Springer -Verlag, New York.

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