Jour, Pure Math
Vol 6 (1987—88) PP 89—91

A NOTE ON BURCHNALL'S OPERATIONAL FORMULA FOR HERMITE POLYNOMIALS

S. P. CHAKRABARTY

In 1941 J. L. Burchnall [1] obtained the following remarkable operational formula for the Hermite polynomials

$$(1)\quad (2x-D)^n y = \sum_{r=0}^{n} (-1)^r \binom{n}{r} H_{n-r}(x)\, D^r y.$$

Later, in 1967, R. M. Wilcox [2] obtained the following general operational formula

$$(2)\quad (P+Q)^n = \sum_{s=0}^{[n/2]} \sum_{k=0}^{n-2s} (\tfrac{1}{2}c)^s\, n!Q^k P^{n-2s-k} \{s!\,k!\,(n-2s-k)!\}^{-1}$$

where the commutator of operators P and Q denoted by [P, Q] satisfies

$[P, Q] \equiv PQ - QP = cI$, c being a number and I the unit operator.

Now if we use $P = -D$, and $Q = 2x$ then $[P, Q] = -2I$ and therefore by (2) we have

$$(3)\quad (2x-D)^n y =$$

$$\sum_{s=0}^{[n/2]} \sum_{m=0}^{n-2s} (-1)^s\, n!\{s!\,m!\,(n-2s-m)!\}^{-1} (2x)^m (-D)^{n-2s-m} y.$$

Now we shall show the equivalence of the results (1) and (3). Indeed if we use the series manipulations viz. ,

$$(4)\quad \sum_{m=o}^{n}\ \sum_{s=o}^{[m/2]} a_m b_s = \sum_{s=o}^{[n/2]}\ \sum_{m=o}^{n-2s} a_{m+2s}\, b_s$$

we get from (3)

$$(2x-D)^n y = \sum_{m=o}^{n}\ \sum_{s=o}^{[m/2]} (-1)^{n-m} \binom{n}{m} (-1)^s\, m!\, \{(m-2s)!\, s!\}^{-1} (2x)^{m-2s} D^{n-m} y$$

$$= \sum_{m=o}^{n} (-1)^{n-m} \binom{n}{m} H_m(x)\, D^{n-m} y$$

$$= \sum_{r=o}^{n} (-1)^r \binom{n}{r} H_{n-r}(x)\, D^r y \text{ (using } n-m = r \text{),}$$

which is exactly Burchnall's formula (1).

Next we consider the formula

$$(5)\quad H_{n+m}(x) = m!\, n! \sum_{r=o}^{\min(m,\, n)} \frac{(-2)^r}{(m-r)!\,(n-r)!\,r!} H_{m-r}(x)\, H_{n-r}(x).$$

Although Burchnall proved the formula (5) from (1), we like to prove (5) directly by employing Wilcox formula (2). In fact we notice that $H_{m+n}(x) = (2x-D)^n H_m(x)$

$$= \sum_{s=o}^{[n/2]}\ \sum_{k=o}^{n-2s} (-1)^s\, n!\, \{s!\, k!\, (n\text{-}2s\text{-}k)!\}^{-1} (2x)^k (-D)^{n-2s-k} H_m(x)$$

i.e. $$= \sum_{s=o}^{[n/2]}\ \sum_{k=o}^{n-2s} n!\, 2^{n-2s} (-1)^{n-2s-k} \binom{m}{m-2s-k} x^k H_{m-n+2s+k}(x),\ n \geq m.$$

Again if we use the series manipulation (4) we obtain

$$H_{m+n}(x) = n! \sum_{k=o}^{n}\ \sum_{s=o}^{[k/2]} \frac{2^{n-2s}(-1)^{n+s-k}}{s!\,(k-2s)!} \binom{m}{n\text{-}k} x^{k-2s} H_{m-n+k}(x)$$

$$= n! \sum_{k=o}^{n} \frac{H_{m-n+k}(x)}{k!} (-2)^{n-k} \binom{m}{n-k} H_k(x).$$

Now using $n-k = r$ we get

$$H_{m+n}(x) = \sum_{r=0}^{n} (-2)^r \binom{m}{r} \binom{n}{r} H_{m-r}(x) H_{n-r}(x) \quad \text{when } n \leq m'$$

which implies (5).

I am indebted to Prof. S. K. Chatterjea of Calcutta University, Department of Pure Mathematics for his kind help in the preparation of this work.

REFERENCES

[1] J. L. Burchnall—Quart. J. Math., (Oxford) 12 (1941), 9-11

[2] Wilcox, R. M., : J, Math. Phys, 8, (1967) 962.

Department of Pure Mathematics
University of Calcutta
Calcutta-700 019
India

Received
31-12-1988